

# The Hamilton-Waterloo Problem for $C_3$ -factors and $C_n$ -factors

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## Abstract

The Hamilton-Waterloo problem asks for a 2-factorization of  $K_v$  (for  $v$  odd) or  $K_v$  minus a 1-factor (for  $v$  even) into  $C_m$ -factors and  $C_n$ -factors. We completely solve the Hamilton-Waterloo problem in the case of  $C_3$ -factors and  $C_n$ -factors for  $n = 4, 5, 7$ .

**Key words:** Hamilton-Waterloo Problem; cycle decomposition; 2-factorization

## 1 Introduction

In this paper, the vertex set and the edge set of a graph  $H$  will be denoted by  $V(H)$  and  $E(H)$ , respectively. We denote the cycle of length  $k$  by  $C_k$ , the complete graph on  $v$  vertices by  $K_v$ , and the complete  $u$ -partite graph with  $u$  parts of size  $g$  by  $K_u[g]$ . A *factor* of a graph  $H$  is a spanning subgraph of  $H$ . Suppose  $G$  is a subgraph of a graph  $H$ , a  $G$ -*factor* of  $H$  is a set of edge-disjoint subgraphs of  $H$ , each isomorphic to  $G$ . And a  $G$ -*factorization* of  $H$  is a set of edge-disjoint  $G$ -factors of  $H$ . Many authors [2, 4, 15, 16, 18, 19, 25, 26] have contributed to prove the following result.

**Theorem 1.1.** *There exists a  $C_k$ -factorization of  $K_u[g]$  if and only if  $g(u-1) \equiv 0 \pmod{2}$ ,  $gu \equiv 0 \pmod{k}$ ,  $k$  is even when  $u = 2$ , and  $(k, u, g) \notin \{(3, 3, 2), (3, 6, 2), (3, 3, 6), (6, 2, 6)\}$ .*

An  $r$ -*factor* is a factor which is  $r$ -regular. It's obvious that a 2-factor consists of a collection of disjoint cycles. A 2-*factorization* of a graph  $H$  is a partition of  $E(H)$  into 2-factors. The well-known Hamilton-Waterloo problem is the problem of determining whether  $K_v$  (for  $v$  odd) or  $K_v$  minus a 1-factor (for  $v$  even) has a 2-factorization in which there are exactly  $\alpha$   $C_m$ -factors and  $\beta$   $C_n$ -factors. For brevity, we generalize this problem to a general graph  $H$ , and use  $\text{HW}(H; m, n, \alpha, \beta)$  to denote a 2-factorization of  $H$  in which there are exactly  $\alpha$   $C_m$ -factors and  $\beta$   $C_n$ -factors. So when  $H = K_v$  (for  $v$  odd) or  $K_v$  minus a 1-factor (for  $v$  even),

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\*Research supported by the National Natural Science Foundation of China under Grant 11571179, the Natural Science Foundation of Jiangsu Province under Grant No. BK20131393, and the Priority Academic Program Development of Jiangsu Higher Education Institutions. E-mail: caohaitao@njnu.edu.cn.

an  $\text{HW}(H; m, n, \alpha, \beta)$  is a solution to the original Hamilton-Waterloo problem, denoted by  $\text{HW}(v; m, n, \alpha, \beta)$ . For convenience, we denote by  $\text{HWP}(v; m, n)$  the set of  $(\alpha, \beta)$  for which a solution  $\text{HW}(v; m, n, \alpha, \beta)$  exists.

It is easy to see that the necessary conditions for the existence of an  $\text{HW}(v; m, n, \alpha, \beta)$  are  $m|v$  when  $\alpha > 0$ ,  $n|v$  when  $\beta > 0$  and  $\alpha + \beta = \lfloor \frac{v-1}{2} \rfloor$ . When  $\alpha\beta = 0$ , the existence of an  $\text{HW}(v; m, n, \alpha, \beta)$  has been solved completely, see Theorem 1.1. From now on, we suppose that  $\alpha\beta \neq 0$ . A lot of work has been done for small values of  $m$  and  $n$ , especially for  $m = 3$ . Adams et al. [1] solved the case  $(m, n) = (3, 5)$  when  $v$  is odd with an exception and some possible exceptions. Danziger [10] and Odabaşı et al. [24] solved the case  $(m, n) = (3, 4)$  with three possible exceptions. Lei et al. [22] solved the case  $(m, n) = (3, 7)$  when  $v$  is odd with three possible exceptions. Asplund et al. [3] focused on  $(m, n) = (3, 3x)$ , and many infinite classes of  $\text{HW}(v; m, n, \alpha, \beta)$ s were constructed. There are also some known results on  $\text{HW}(v; 3, v, \alpha, \beta)$ , see [11, 12, 14, 21]. For more results on Hamilton-Waterloo problem, the reader is refer to [5, 6, 7, 8, 9, 13, 17, 20, 23].

**Theorem 1.2.** ([10, 24])  $(\alpha, \beta) \in \text{HWP}(v; 3, 4)$  if and only if  $v \equiv 0 \pmod{12}$ ,  $\alpha + \beta = \lfloor \frac{v-1}{2} \rfloor$ , except possibly for  $(v, \alpha, \beta) = (24, 5, 6), (24, 9, 2), (48, 17, 6)$ .

**Theorem 1.3.** ([1]) Suppose  $v \equiv 15 \pmod{30}$  and  $\alpha + \beta = \frac{v-1}{2}$ . Then  $(\alpha, \beta) \in \text{HWP}(v; 3, 5)$  except for  $(v, \alpha, \beta) = (15, 6, 1)$ , and except possibly for  $(\alpha, \beta) = (\frac{v-3}{2}, 1)$  when  $v > 15$ .

**Theorem 1.4.** ([22]) Suppose  $v \equiv 21 \pmod{42}$  and  $\alpha + \beta = \frac{v-1}{2}$ . Then  $(\alpha, \beta) \in \text{HWP}(v; 3, 7)$ , except possibly for  $(v, \alpha, \beta) = (21, 2, 8), (21, 4, 6), (21, 6, 4)$ .

Combining the known results in Theorems 1.2-1.4, we will prove the following main result.

**Theorem 1.5.** For  $n \in \{4, 5, 7\}$ ,  $(\alpha, \beta) \in \text{HWP}(v; 3, n)$  if and only if  $v \equiv 0 \pmod{3n}$ ,  $\alpha + \beta = \lfloor \frac{v-1}{2} \rfloor$  and  $(v, \alpha, \beta) \neq (15, 6, 1)$ .

## 2 Constructions

Let  $\Gamma$  be a finite additive group and let  $S$  be a subset of  $\Gamma \setminus \{0\}$  such that the opposite of every element of  $S$  also belongs to  $S$ . The *Cayley graph* over  $\Gamma$  with connection set  $S$ , denoted by  $\text{Cay}(\Gamma, S)$ , is the graph with vertex set  $\Gamma$  and edge set  $E(\text{Cay}(\Gamma, S)) = \{(a, b) | a, b \in \Gamma, a - b \in S\}$ . It is quite obvious that  $\text{Cay}(\Gamma, S) = \text{Cay}(\Gamma, \pm S)$ .

**Lemma 2.1.** Let  $n \geq 3$ . If  $a \in Z_n$ , the order of  $a$  is greater than 3 and  $(i, m) = 1$ , then there is a  $C_m$ -factorization of  $\text{Cay}(Z_n \times Z_m, \pm\{0, a, 2a\} \times \{\pm i\})$ .

*Proof:* Since the order of  $a$  is greater than 3, we have  $|\{0, a, -a, 2a, -2a\}| = 5$ . Let  $C_j = ((a_{j0}, b_{j0}) = (0, 0), (a_{j1}, b_{j1}), \dots, (a_{j,m-1}, b_{j,m-1}))$ ,  $1 \leq j \leq 5$ , where

$$\begin{aligned} a_{11} &= a, & a_{21} &= 0, & a_{31} &= 2a, & a_{41} &= -a, & a_{51} &= -2a, \\ a_{12} &= 2a, & a_{22} &= -2a, & a_{32} &= a, & a_{42} &= -a, & a_{52} &= 0, \end{aligned}$$

$$\begin{aligned} a_{jt} &= a_{j,(t-2)}, t \geq 3, \\ b_{jt} &= ti \pmod{m}, 1 \leq t \leq m-1. \end{aligned}$$

Since  $(i, m) = 1$ , we know that  $b_{jt}$ ,  $0 \leq t \leq m-1$ , are all different modulo  $m$ . Then each  $C_j$  will generate a  $C_m$ -factor by  $(+1 \pmod{n}, -)$ . Thus we can obtain the required 5  $C_m$ -factors which form a  $C_m$ -factorization of  $\text{Cay}(Z_n \times Z_m, \pm\{0, a, 2a\} \times \{\pm i\})$ .  $\square$

**Lemma 2.2.** *Let  $n \geq 3$ . If  $a \in Z_n$ , the order of  $a$  is greater than 2 and  $(i, m) = 1$ , then there is a  $C_m$ -factorization of  $\text{Cay}(Z_n \times Z_m, \pm\{0, a\} \times \{\pm i\})$ .*

*Proof:* Because the order of  $a$  is greater than 2, we have  $|\{0, a, -a\}| = 3$ . Let  $C_j = ((a_{j0}, b_{j0}) = (0, 0), (a_{j1}, b_{j1}), \dots, (a_{j,m-1}, b_{j,m-1}))$ ,  $1 \leq j \leq 3$ , where

$$\begin{aligned} a_{11} &= a, a_{21} = 0, a_{31} = -a, \\ a_{12} &= a, a_{22} = -a, a_{32} = 0, \\ a_{jt} &= a_{j,(t-2)}, t \geq 3, \\ b_{jt} &= ti \pmod{m}, 1 \leq t \leq m-1. \end{aligned}$$

Since  $(i, m) = 1$ , we know that  $b_{jt}$ ,  $0 \leq t \leq m-1$ , are all different modulo  $m$ . Then each  $C_j$  will generate a  $C_m$ -factor by  $(+1 \pmod{n}, -)$ . Thus we can obtain the required 3  $C_m$ -factors which form a  $C_m$ -factorization of  $\text{Cay}(Z_n \times Z_m, \pm\{0, a\} \times \{\pm i\})$ .  $\square$

For our recursive constructions, we need the definition of an incomplete Hamilton-Waterloo problem design. Suppose  $G$  is a subgraph of a graph  $H$ . A *holey 2-factor* of  $H - G$  is a 2-regular subgraph of  $H$  covering all vertices except those belonging to  $G$ . We will also frequently speak of a holey  $C_k$ -factor to mean a holey 2-factor whose cycles all have length  $k$ . Let  $v - h \equiv 0 \pmod{2}$ . An *incomplete Hamilton-Waterloo problem design* on  $v$  vertices with a hole of size  $h$ , denoted by  $\text{IHW}(v, h; m, n, \alpha, \beta, \alpha', \beta')$ , is a cycle decomposition of  $K_v - E(K_h)$  if  $v$  is odd, or  $K_v - E(K_h)$  minus a 1-factor  $I$  if  $v$  is even, such that (1)  $\alpha + \beta = \frac{v-h}{2}$ ,  $\alpha' + \beta' = \lfloor \frac{h-1}{2} \rfloor$ ; (2) there are  $\alpha$   $C_m$ -factors and  $\beta$   $C_n$ -factors of  $K_v$ ; (3) there are  $\alpha'$  holey  $C_m$ -factors and  $\beta'$  holey  $C_n$ -factors of  $K_v - K_h$ . We denote by  $\text{IHWP}(v, h; m, n)$  the set of  $(\alpha, \beta, \alpha', \beta')$  for which an  $\text{IHW}(v, h; m, n, \alpha, \beta, \alpha', \beta')$  exists.

**Lemma 2.3.**  $(15, 0, 6, 1) \in \text{IHWP}(45, 15; 3, 5)$ .

*Proof:* Let the vertex set be  $(Z_6 \times Z_5) \cup \{\infty_{i=0}^{14}\}$ . A holey  $C_5$ -factor is  $\text{Cay}(Z_6 \times Z_5, \{0\} \times \{\pm 2\})$ . The required 15  $C_3$ -factors will be generated from three initial  $C_3$ -factors  $P_i$  ( $i = 1, 2, 3$ ) by  $(-, +1 \pmod{5})$ . For the required six holey  $C_3$ -factors, five of which can be generated from an initial holey  $C_3$ -factor  $Q$  by  $(-, +1 \pmod{5})$ . The last holey  $C_3$ -factor can be generated from two base cycles  $(0_0, 3_4, 2_0)$  and  $(1_1, 5_1, 4_2)$  by  $(-, +1 \pmod{5})$ . The cycles of  $P_i$  and  $Q$  are listed below.

$P_1$	$(\infty_0, 4_4, 5_0)$ $(\infty_6, 3_4, 2_4)$ $(\infty_{12}, 0_1, 3_1)$	$(\infty_1, 1_1, 3_3)$ $(\infty_7, 0_2, 1_0)$ $(\infty_{13}, 2_0, 5_4)$	$(\infty_2, 0_0, 2_2)$ $(\infty_8, 1_2, 5_3)$ $(\infty_{14}, 1_3, 2_1)$	$(\infty_3, 4_0, 5_2)$ $(\infty_9, 1_4, 0_4)$ $(\infty_{10}, 4_2, 3_2)$	$(\infty_4, 2_3, 3_0)$ $(\infty_{10}, 4_2, 3_2)$ $(\infty_{11}, 0_3, 4_3)$	$(\infty_5, 5_1, 4_1)$ $(\infty_{11}, 0_3, 4_3)$ $(\infty_{12}, 2_0, 5_4)$
$P_2$	$(\infty_0, 0_0, 1_1)$ $(\infty_6, 4_0, 1_3)$ $(\infty_{12}, 2_0, 4_3)$	$(\infty_1, 2_2, 4_4)$ $(\infty_7, 2_4, 5_2)$ $(\infty_{13}, 3_1, 0_4)$	$(\infty_2, 3_3, 5_0)$ $(\infty_8, 3_0, 4_2)$ $(\infty_{14}, 5_3, 3_2)$	$(\infty_3, 0_1, 3_4)$ $(\infty_9, 4_1, 5_4)$ $(\infty_{10}, 0_3, 1_0)$	$(\infty_4, 1_2, 5_1)$ $(\infty_{10}, 0_3, 1_0)$ $(\infty_{11}, 1_4, 2_1)$	$(\infty_5, 2_3, 0_2)$ $(\infty_{11}, 1_4, 2_1)$ $(\infty_{12}, 2_0, 4_3)$
$P_3$	$(\infty_0, 2_2, 3_3)$ $(\infty_6, 5_1, 0_2)$ $(\infty_{12}, 5_3, 1_0)$	$(\infty_1, 0_0, 5_0)$ $(\infty_7, 3_0, 4_1)$ $(\infty_{13}, 1_4, 4_3)$	$(\infty_2, 1_1, 4_4)$ $(\infty_8, 0_3, 2_1)$ $(\infty_{14}, 4_2, 0_4)$	$(\infty_3, 1_2, 2_3)$ $(\infty_9, 2_4, 3_2)$ $(\infty_{10}, 5_2, 2_0)$	$(\infty_4, 0_1, 4_0)$ $(\infty_{10}, 5_2, 2_0)$ $(\infty_{11}, 3_1, 5_4)$	$(\infty_5, 3_4, 1_3)$ $(\infty_{11}, 3_1, 5_4)$ $(\infty_{12}, 5_3, 1_0)$
$Q$	$(0_0, 0_1, 4_2)$ $(1_2, 3_0, 5_4)$	$(1_1, 4_1, 2_0)$ $(2_3, 1_3, 0_4)$	$(2_2, 5_2, 2_1)$ $(3_4, 1_4, 1_0)$	$(3_3, 0_2, 5_3)$ $(4_0, 3_1, 3_2)$	$(4_4, 2_4, 4_3)$ $(5_0, 5_1, 0_3)$	$(5_0, 5_1, 0_3)$ $(5_0, 5_1, 0_3)$

□

For next recursive construction, we still need the definition of a cycle frame. Let  $H$  be a graph  $K_u[g]$  with  $u$  parts  $G_1, G_2, \dots, G_u$ . A partition of  $E(H)$  into holey 2-factors of  $H - G_i (1 \leq i \leq u)$  is said to be a *cycle frame of type  $g^u$* . Further, if all holey 2-factors of a cycle frame of type  $g^u$  are  $C_k$ -factors, then we denote the cycle frame by  $k\text{-CF}(g^u)$ .

**Theorem 2.4.** ([27]) *There exists a  $3\text{-CF}(g^u)$  if and only if  $g \equiv 0 \pmod{2}$ ,  $g(u-1) \equiv 0 \pmod{3}$  and  $u \geq 4$ .*

It's obvious that there are exactly  $\frac{g}{2}$  holey 2-factors with respect to each part. We use  $\text{CF}(g^u; m, n, \alpha, \beta)$  with  $\alpha + \beta = \frac{g}{2}$  to denote a cycle frame of type  $g^u$  in which there are exactly  $\alpha$  holey  $C_m$ -factors and  $\beta$  holey  $C_n$ -factors with respect to each part. Now we use cycle frames and incomplete Hamilton-Waterloo problem designs to give the “Filling in Holes” construction.

**Construction 2.5.** *Let  $\alpha + \beta = \frac{g}{2}$ ,  $\alpha' + \beta' = \lfloor \frac{h-1}{2} \rfloor$ . If there exist a  $\text{CF}(g^u; m, n, \alpha, \beta)$ , an  $\text{IHW}(g+h, h; m, n, \alpha, \beta, \alpha', \beta')$  and an  $\text{HW}(g+h; m, n, \alpha + \alpha', \beta + \beta')$ , then an  $\text{HW}(gu+h; m, n, \alpha u + \alpha', \beta u + \beta')$  exists.*

*Proof:* We start with a  $\text{CF}(g^u; m, n, \alpha, \beta)$ , for each part  $G_i$ ,  $1 \leq i \leq u$ , denote its  $\alpha$  holey  $C_m$ -factors by  $P_{ij} (1 \leq j \leq \alpha)$ , and denote its  $\beta$  holey  $C_n$ -factors by  $Q_{ij} (1 \leq j \leq \beta)$ .

For each  $i (1 \leq i \leq u-1)$ , place a copy of an  $\text{IHW}(g+h, h; m, n, \alpha, \beta, \alpha', \beta')$  on the vertices of the part  $G_i$  and  $h$  new common vertices (take the subgraph on these  $h$  vertices as the hole), whose  $\alpha$   $C_m$ -factors and  $\beta$   $C_n$ -factors are denoted by  $P'_{ij} (1 \leq j \leq \alpha)$  and  $Q'_{ij} (1 \leq j \leq \beta)$  respectively,  $\alpha'$  holey  $C_m$ -factors and  $\beta'$  holey  $C_n$ -factors are denoted by  $P''_{ij} (1 \leq j \leq \alpha')$  and  $Q''_{ij} (1 \leq j \leq \beta')$  respectively. Further, if  $h \equiv 0 \pmod{2}$ , then  $g+h \equiv 0 \pmod{2}$  (note that the existence of a  $\text{CF}(g^u)$  requires  $g \equiv 0 \pmod{2}$ ). Then according to the definition of an  $\text{IHW}$ , there is a 1-factor  $I_i$  of the subgraph on the vertices from  $G_i$ .

Place on the vertices of the part  $G_u$  and these  $h$  common vertices a copy of an  $\text{HW}(g+h; m, n, \alpha + \alpha', \beta + \beta')$  with  $\alpha + \alpha'$   $C_m$ -factors  $P'_{uj} (1 \leq j \leq \alpha + \alpha')$  and  $\beta + \beta'$   $C_n$ -factors  $Q'_{uj} (1 \leq j \leq \beta + \beta')$ . If  $h \equiv 0 \pmod{2}$ , there is a 1-factor  $I_u$ .

Let

$$\begin{aligned} S_{ij} &= P_{ij} \cup P'_{ij}, 1 \leq i \leq u, 1 \leq j \leq \alpha, \\ F_{ij} &= Q_{ij} \cup Q'_{ij}, 1 \leq i \leq u, 1 \leq j \leq \beta, \\ S_{u,j+\alpha} &= (\cup_{i=1}^{u-1} P''_{ij}) \cup P'_{u,j+\alpha}, 1 \leq j \leq \alpha', \\ F_{u,j+\beta} &= (\cup_{i=1}^{u-1} Q''_{ij}) \cup Q'_{u,j+\beta}, 1 \leq j \leq \beta'. \end{aligned}$$

Then both  $S_{ij}$  and  $S_{u,j+\alpha}$  are  $C_m$ -factors,  $F_{ij}$  and  $F_{u,j+\beta}$  are  $C_n$ -factors on the whole vertex set, and they form an  $\text{HW}(gu+h; m, n, \alpha u + \alpha', \beta u + \beta')$ . Note that if  $h \equiv 0 \pmod{2}$ ,  $I = \cup_{i=1}^u I_i$  is a 1-factor on the whole vertex set.  $\square$

For the next recursive construction, we need more notations. When  $g(u-1) \equiv 1 \pmod{2}$ , by Theorem 1.1 it is easy to see that an  $\text{HW}(K_u[g]; m, n, \alpha, \beta)$  can not exist. In this case, by simple computation, we know that it is possible to partition  $E(K_u[g])$  into a 1-factor,  $\alpha$   $C_m$ -factors and  $\beta$   $C_n$ -factors, where  $\alpha + \beta = \lfloor \frac{g(u-1)}{2} \rfloor$ . For brevity, we still use  $\text{HW}(K_u[g]; m, n, \alpha, \beta)$  to denote such a decomposition.

**Construction 2.6.** Suppose there exist an  $\text{HW}(K_u[g]; m, n, \alpha, \beta)$  and an  $\text{HW}(g; m, n, \alpha', \beta')$ , then an  $\text{HW}(gu; m, n, \alpha + \alpha', \beta + \beta')$  exists.

*Proof:* We start with an  $\text{HW}(K_u[g]; m, n, \alpha, \beta)$  whose  $\alpha$   $C_m$ -factors are denoted by  $P_j (1 \leq j \leq \alpha)$ ,  $\beta$   $C_n$ -factors are denoted by  $Q_j (1 \leq j \leq \beta)$ , and a 1-factor (when  $g(u-1) \equiv 1 \pmod{2}$ ) is denoted by  $I$ .

For each  $i (1 \leq i \leq u)$ , place a copy of an  $\text{HW}(g; m, n, \alpha', \beta')$  on the vertices of the part  $G_i$  whose  $\alpha'$   $C_m$ -factors and  $\beta'$   $C_n$ -factors are denoted by  $P'_{ij} (1 \leq j \leq \alpha')$  and  $Q'_{ij} (1 \leq j \leq \beta')$  respectively, and a 1-factor is denoted by  $I_i$  if  $g \equiv 0 \pmod{2}$ . Let  $S_j = \cup_{i=1}^u P'_{ij} (1 \leq j \leq \alpha')$  and  $F_j = \cup_{i=1}^u Q'_{ij} (1 \leq j \leq \beta')$ . Then  $S_j$  is a  $C_m$ -factor and  $F_j$  is a  $C_n$ -factor of the required  $\text{HW}(gu; m, n, \alpha + \alpha', \beta + \beta')$ . So we have obtained  $\alpha + \alpha'$   $C_m$ -factors and  $\beta + \beta'$   $C_n$ -factors. At last,  $\cup_{i=1}^u I_i$  is a 1-factor if  $g \equiv 0 \pmod{2}$  and  $I$  is a 1-factor if  $g \equiv 1 \pmod{2}$  and  $u \equiv 0 \pmod{2}$ .  $\square$

For the next construction, we need the definition of lexicographic product of two graphs. Given a graph  $G$ ,  $G[n]$  is the *lexicographic product* of  $G$  with the empty graph on  $n$  points. Specifically, the vertex set is  $\{x_i : x \in V(G), i \in Z_n\}$  and  $x_i y_j \in E(G[n])$  if and only if  $xy \in E(G), i, j \in Z_n$ . In the following we will denote by  $C_m[n]$  the lexicographic product of  $C_m$  with the empty graph on  $n$  points.

**Construction 2.7.** If  $(\alpha, \beta) \in \text{HWP}(K_u[g]; m, n)$ ,  $(t_i, s - t_i) \in \text{HWP}(C_m[s]; m', n')$ ,  $1 \leq i \leq \alpha$ , and  $(r_j, s - r_j) \in \text{HWP}(C_n[s]; m', n')$ ,  $1 \leq j \leq \beta$ , then  $(\alpha', \beta') \in \text{HWP}(K_u[gs]; m', n')$ , where  $\alpha' = \sum_{i=1}^{\alpha} t_i + \sum_{j=1}^{\beta} r_j$  and  $\beta' = (\alpha + \beta)s - \alpha'$ .

*Proof:* We start with an  $\text{HW}(K_u[g]; m, n, \alpha, \beta)$  with  $\alpha$   $C_m$ -factors and  $\beta$   $C_n$ -factors. Give each vertex weight  $s$ , then we obtain  $\alpha$   $C_m[s]$ -factors and  $\beta$   $C_n[s]$ -factors. Now we replace each  $C_m[s]$  and each  $C_n[s]$  in the  $i$ -th  $C_m[s]$ -factor and the  $j$ -th  $C_n[s]$ -factor with an  $\text{HW}(C_m[s]; m', n', t_i, s - t_i)$  and an  $\text{HW}(C_n[s]; m', n', r_j, s - r_j)$  respectively. Further, take one of the  $t_i$   $C_{m'}$ -factors from each  $\text{HW}(C_m[s]; m', n', t_i, s - t_i)$  in the  $i$ -th  $C_m[s]$ -factor, and put them together to get a  $C_{m'}$ -factor of  $K_u[gs]$ . Thus, we have obtained  $\sum_{i=1}^{\alpha} t_i$   $C_{m'}$ -factor of  $K_u[gs]$ . Similarly, we can get  $\sum_{j=1}^{\beta} r_j$   $C_{m'}$ -factor of  $K_u[gs]$  from the known  $\text{HW}(C_n[s]; m', n', r_j, s - r_j)$ , and  $\sum_{i=1}^{\alpha} (s - t_i) + \sum_{j=1}^{\beta} (s - r_j) = (\alpha + \beta)s - \alpha'$   $C_{n'}$ -factors of  $K_u[gs]$ .  $\square$

### 3 HWP( $v; 3, 4$ )

In this section, we will give three direct constructions and complete the spectrum for an  $\text{HW}(v; 3, 4, \alpha, \beta)$ .

**Lemma 3.1.**  $(9, 2) \in \text{HWP}(24; 3, 4)$ .

*Proof:* Let the vertex set be  $\Gamma = Z_8 \times Z_3$ , and the 1-factor be  $\text{Cay}(\Gamma, \{4\} \times \{0\})$ . For the 9  $C_3$ -factors, let  $F = \{Q, Q + 4_0\}$ , where  $Q = \{(0_0, 6_0, 0_1), (1_1, 1_0, 3_1), (2_2, 2_1, 7_0), (5_2, 3_2, 4_2)\}$ . It's easy to see that  $F$  is a  $C_3$ -factor since all these 4 elements having the same subscript in  $Q$  are different modulo 4. Then  $F, F + 4_1, F + 0_2$  are 3  $C_3$ -factors. The other 6  $C_3$ -factors can be generated from an initial  $C_3$ -factor  $P = \{(0_0, 1_1, 2_2), (3_0, 4_1, 7_1), (5_2, 0_2, 4_0), (6_0, 1_0, 5_1), (2_1, 3_2, 3_1), (6_2, 2_0, 4_2), (7_0, 1_2, 6_1), (0_1, 5_0, 7_2)\}$  by  $(+4 \pmod{8}, +1 \pmod{3})$ .

The required two  $C_4$ -factors can be generated from two base 4-cycles  $(0_0, 2_1, 5_0, 3_2)$  and  $(0_0, 5_1, 6_1, 3_1)$  by  $(+4 \pmod{8}, +1 \pmod{3})$  since the first coordinate of the four elements in each cycle are different modulo 4.  $\square$

**Lemma 3.2.**  $(5, 6) \in \text{HWP}(24; 3, 4)$ .

*Proof:* Let  $\Gamma = Z_8 \times Z_3$ . Firstly, we construct an  $\text{HW}(K_3[8]; 3, 4, 5, 3)$  with three parts  $Z_8 \times \{i\}$ ,  $i \in Z_3$ . The required 5  $C_3$ -factors come from a  $C_3$ -factorization of  $\text{Cay}(\Gamma, \pm\{0, 1, 2\} \times \{\pm 1\})$  by Lemma 2.1. The required three  $C_4$ -factors will be generated from an initial  $C_4$ -factor  $P = \{(0_0, 4_1, 0_2, 5_1), (1_1, 4_0, 7_1, 4_2), (2_2, 6_0, 1_2, 5_0), (3_0, 0_1, 3_2, 6_1), (5_2, 1_0, 6_2, 2_0), (2_1, 7_0, 3_1, 7_2)\}$  by  $(-, +1 \pmod{3})$ . Then we use Construction 2.6 with an  $\text{HW}(8; 3, 4, 0, 3)$  from Theorem 1.1 and an  $\text{HW}(K_3[8]; 3, 4, 5, 3)$  constructed above to get an  $\text{HW}(24; 3, 4, 5, 6)$ .  $\square$

**Lemma 3.3.**  $(17, 6) \in \text{HWP}(48; 3, 4)$ .

*Proof:* Let the vertex set be  $\Gamma = Z_{16} \times Z_3$ , and the 1-factor be  $\text{Cay}(\Gamma, \{8\} \times \{0\})$ . The required 5 of 17  $C_3$ -factors come from a  $C_3$ -factorization of  $\text{Cay}(\Gamma, \pm\{0, 1, 2\} \times \{\pm 1\})$  by

Lemma 2.1. The other 12  $C_3$ -factors can be generated from an initial  $C_3$ -factor  $P$  by  $(+4 \pmod{16}, +1 \pmod{3})$ . For the required 6  $C_4$ -factors, start with a cycle set  $Q$  in which all these 8 elements having the same subscript are different modulo 8. Let  $F = \{Q, Q + 8_0\}$ . Then  $F, F + 4_1, F + 8_2, F + 12_0, F + 0_1, F + 4_2$  are 6  $C_4$ -factors. The cycles of  $P$  and  $Q$  are listed below.

$P$	$(0_0, 3_0, 6_0)$	$(1_1, 4_1, 8_2)$	$(2_2, 5_2, 9_0)$	$(7_1, 11_2, 0_1)$	$(10_1, 14_2, 3_1)$	$(12_0, 1_2, 6_1)$
	$(13_1, 5_0, 15_1)$	$(15_0, 7_2, 1_0)$	$(2_0, 14_0, 8_1)$	$(4_2, 0_2, 9_2)$	$(8_0, 5_1, 11_1)$	$(9_1, 4_0, 15_2)$
	$(10_2, 7_0, 13_0)$	$(11_0, 2_1, 10_0)$	$(12_1, 3_2, 14_1)$	$(13_2, 6_2, 12_2)$		
$Q$	$(0_0, 8_2, 3_2, 15_0)$	$(1_1, 14_2, 8_1, 4_2)$	$(2_2, 1_2, 11_1, 15_1)$			
	$(3_0, 12_1, 5_2, 14_0)$	$(9_0, 5_0, 10_1, 5_1)$	$(12_0, 10_0, 6_1, 15_2)$			

□

Combining Theorem 1.2, Lemmas 3.1, 3.2 and 3.3, we have the following theorem.

**Theorem 3.4.**  $(\alpha, \beta) \in \text{HWP}(v; 3, 4)$  if and only if  $v \equiv 0 \pmod{12}$  and  $\alpha + \beta = \lfloor \frac{v-1}{2} \rfloor$ .

## 4 $\text{HWP}(v; 3, 5)$

In this section, we shall solve the left infinite class in [1] for the existence of an  $\text{HW}(v; 3, 5, \alpha, \beta)$  when  $v \equiv 15 \pmod{30}$ . Then we continue to consider the existence of an  $\text{HW}(v; 3, 5, \alpha, \beta)$  when  $v \equiv 0 \pmod{30}$ .

**Lemma 4.1.**  $(21, 1) \in \text{HWP}(45; 3, 5)$ .

*Proof:* Let the vertex set be  $\Gamma = Z_9 \times Z_5$ . The required  $C_5$ -factor is  $\text{Cay}(\Gamma, \{0\} \times \{\pm 1\})$ . For the required 21  $C_3$ -factors, 15 of which will be generated from an initial  $C_3$ -factor  $P$  by  $(+3 \pmod{9}, +1 \pmod{5})$ . Each cycle in  $Q$  will generate a  $C_3$ -factor by  $(+3 \pmod{9}, +1 \pmod{5})$  since the 3 elements in the first coordinate are different modulo 3. Thus we have obtained the last 6  $C_3$ -factors. The cycles of  $P$  and  $Q$  are listed below.

$P$	$(1_1, 4_1, 7_4)$	$(4_2, 0_1, 3_4)$	$(7_2, 8_2, 7_0)$	$(8_3, 0_2, 4_0)$	$(6_1, 6_3, 7_3)$	$(2_2, 5_2, 8_4)$	$(3_2, 7_1, 5_1)$	$(3_3, 1_0, 5_4)$
	$(0_4, 3_1, 8_1)$	$(6_4, 3_0, 2_3)$	$(4_4, 5_3, 1_3)$	$(4_3, 1_4, 2_0)$	$(0_0, 6_0, 8_0)$	$(5_0, 2_1, 2_4)$	$(0_3, 1_2, 6_2)$	
$Q$	$(0_0, 1_1, 5_2)$	$(0_0, 2_2, 7_0)$	$(0_0, 7_1, 8_4)$	$(0_0, 4_2, 2_3)$	$(0_0, 5_3, 7_4)$	$(0_0, 5_1, 7_3)$		

□

**Lemma 4.2.**  $(\frac{v-3}{2}, 1) \in \text{HWP}(v; 3, 5)$  for  $v = 75, 105$ .

*Proof:* Let  $v = 3u$  and the vertex set be  $\Gamma = Z_u \times Z_3$ . The required  $C_5$ -factor is  $\text{Cay}(\Gamma, \{\pm 10\} \times \{0\})$  when  $v = 75$  or  $\text{Cay}(\Gamma, \{\pm 7\} \times \{0\})$  when  $v = 105$ .

For the required  $\frac{3(u-1)}{2}$   $C_3$ -factors,  $u$  of which will be generated from an initial  $C_3$ -factor  $P$  by  $(+1 \pmod{u}, -)$ . The other  $\frac{u-3}{2}$   $C_3$ -factors will be obtained from  $\frac{u-3}{2}$  3-cycles in  $Q$ . Each cycle of  $Q$  will generate a  $C_3$ -factor by  $(+1 \pmod{u}, -)$  since the first coordinate of those 3 elements of the cycle are different modulo 3. The cycles of  $P$  and  $Q$  for each  $v$  are listed below.

$v = 75$  :

$P$	(9 <sub>0</sub> , 18 <sub>0</sub> , 5 <sub>0</sub> )	(6 <sub>1</sub> , 13 <sub>2</sub> , 0 <sub>2</sub> )	(4 <sub>1</sub> , 17 <sub>0</sub> , 18 <sub>1</sub> )	(22 <sub>1</sub> , 5 <sub>1</sub> , 7 <sub>0</sub> )	(5 <sub>2</sub> , 23 <sub>0</sub> , 3 <sub>2</sub> )	(0 <sub>0</sub> , 10 <sub>1</sub> , 19 <sub>0</sub> )
	(17 <sub>2</sub> , 12 <sub>2</sub> , 18 <sub>2</sub> )	(1 <sub>1</sub> , 8 <sub>0</sub> , 13 <sub>0</sub> )	(2 <sub>2</sub> , 19 <sub>2</sub> , 24 <sub>1</sub> )	(6 <sub>0</sub> , 8 <sub>2</sub> , 4 <sub>0</sub> )	(13 <sub>1</sub> , 19 <sub>1</sub> , 4 <sub>2</sub> )	(0 <sub>1</sub> , 9 <sub>1</sub> , 21 <sub>1</sub> )
	(2 <sub>0</sub> , 2 <sub>1</sub> , 24 <sub>2</sub> )	(7 <sub>1</sub> , 20 <sub>2</sub> , 1 <sub>0</sub> )	(3 <sub>1</sub> , 11 <sub>0</sub> , 21 <sub>2</sub> )	(11 <sub>2</sub> , 14 <sub>2</sub> , 16 <sub>0</sub> )	(1 <sub>2</sub> , 22 <sub>2</sub> , 15 <sub>2</sub> )	(21 <sub>0</sub> , 14 <sub>0</sub> , 22 <sub>0</sub> )
	(23 <sub>2</sub> , 12 <sub>1</sub> , 11 <sub>1</sub> )	(16 <sub>1</sub> , 6 <sub>2</sub> , 23 <sub>1</sub> )	(12 <sub>0</sub> , 15 <sub>0</sub> , 14 <sub>1</sub> )	(7 <sub>2</sub> , 16 <sub>2</sub> , 20 <sub>0</sub> )	(3 <sub>0</sub> , 8 <sub>1</sub> , 9 <sub>2</sub> )	(24 <sub>0</sub> , 10 <sub>2</sub> , 10 <sub>0</sub> )
	(15 <sub>1</sub> , 17 <sub>1</sub> , 20 <sub>1</sub> )					
$Q$	(0 <sub>0</sub> , 4 <sub>1</sub> , 8 <sub>2</sub> )	(0 <sub>0</sub> , 7 <sub>1</sub> , 16 <sub>2</sub> )	(0 <sub>0</sub> , 14 <sub>2</sub> , 8 <sub>1</sub> )	(0 <sub>0</sub> , 17 <sub>2</sub> , 3 <sub>1</sub> )	(0 <sub>0</sub> , 19 <sub>1</sub> , 15 <sub>2</sub> )	(0 <sub>0</sub> , 22 <sub>1</sub> , 24 <sub>2</sub> )
	(0 <sub>0</sub> , 1 <sub>2</sub> , 21 <sub>1</sub> )	(0 <sub>0</sub> , 9 <sub>1</sub> , 9 <sub>2</sub> )	(0 <sub>0</sub> , 13 <sub>2</sub> , 14 <sub>1</sub> )	(0 <sub>0</sub> , 3 <sub>2</sub> , 11 <sub>1</sub> )	(0 <sub>0</sub> , 18 <sub>2</sub> , 20 <sub>1</sub> )	

$v = 105$  :

$P$	(3 <sub>1</sub> , 29 <sub>1</sub> , 28 <sub>2</sub> )	(9 <sub>0</sub> , 10 <sub>0</sub> , 26 <sub>0</sub> )	(1 <sub>0</sub> , 24 <sub>0</sub> , 28 <sub>1</sub> )	(5 <sub>1</sub> , 24 <sub>1</sub> , 26 <sub>1</sub> )	(0 <sub>2</sub> , 2 <sub>2</sub> , 29 <sub>2</sub> )	(21 <sub>0</sub> , 31 <sub>0</sub> , 9 <sub>1</sub> )
	(30 <sub>0</sub> , 32 <sub>0</sub> , 32 <sub>1</sub> )	(2 <sub>1</sub> , 19 <sub>1</sub> , 17 <sub>2</sub> )	(27 <sub>0</sub> , 25 <sub>2</sub> , 34 <sub>2</sub> )	(10 <sub>1</sub> , 30 <sub>1</sub> , 14 <sub>2</sub> )	(0 <sub>1</sub> , 6 <sub>1</sub> , 16 <sub>2</sub> )	(6 <sub>0</sub> , 11 <sub>2</sub> , 12 <sub>2</sub> )
	(7 <sub>1</sub> , 18 <sub>1</sub> , 3 <sub>2</sub> )	(4 <sub>1</sub> , 34 <sub>1</sub> , 1 <sub>2</sub> )	(15 <sub>0</sub> , 29 <sub>0</sub> , 14 <sub>1</sub> )	(4 <sub>2</sub> , 23 <sub>2</sub> , 27 <sub>2</sub> )	(0 <sub>0</sub> , 10 <sub>2</sub> , 15 <sub>2</sub> )	(8 <sub>0</sub> , 19 <sub>0</sub> , 23 <sub>0</sub> )
	(14 <sub>0</sub> , 5 <sub>2</sub> , 22 <sub>2</sub> )	(3 <sub>0</sub> , 25 <sub>0</sub> , 15 <sub>1</sub> )	(4 <sub>0</sub> , 7 <sub>0</sub> , 13 <sub>0</sub> )	(18 <sub>2</sub> , 21 <sub>2</sub> , 32 <sub>2</sub> )	(2 <sub>0</sub> , 20 <sub>2</sub> , 30 <sub>2</sub> )	(1 <sub>1</sub> , 23 <sub>1</sub> , 33 <sub>1</sub> )
	(12 <sub>0</sub> , 20 <sub>0</sub> , 13 <sub>1</sub> )	(8 <sub>1</sub> , 12 <sub>1</sub> , 20 <sub>1</sub> )	(28 <sub>0</sub> , 33 <sub>0</sub> , 17 <sub>1</sub> )	(21 <sub>1</sub> , 22 <sub>1</sub> , 8 <sub>2</sub> )	(16 <sub>0</sub> , 13 <sub>2</sub> , 33 <sub>2</sub> )	(22 <sub>0</sub> , 9 <sub>2</sub> , 31 <sub>2</sub> )
	(5 <sub>0</sub> , 11 <sub>1</sub> , 24 <sub>2</sub> )	(11 <sub>0</sub> , 16 <sub>1</sub> , 7 <sub>2</sub> )	(17 <sub>0</sub> , 25 <sub>1</sub> , 19 <sub>2</sub> )	(18 <sub>0</sub> , 27 <sub>1</sub> , 6 <sub>2</sub> )	(34 <sub>0</sub> , 31 <sub>1</sub> , 26 <sub>2</sub> )	
$Q$	(0 <sub>0</sub> , 3 <sub>1</sub> , 3 <sub>2</sub> )	(0 <sub>0</sub> , 7 <sub>1</sub> , 0 <sub>2</sub> )	(0 <sub>0</sub> , 10 <sub>1</sub> , 11 <sub>2</sub> )	(0 <sub>0</sub> , 11 <sub>1</sub> , 16 <sub>2</sub> )	(0 <sub>0</sub> , 14 <sub>1</sub> , 21 <sub>2</sub> )	(0 <sub>0</sub> , 15 <sub>1</sub> , 4 <sub>2</sub> )
	(0 <sub>0</sub> , 16 <sub>1</sub> , 24 <sub>2</sub> )	(0 <sub>0</sub> , 17 <sub>1</sub> , 20 <sub>2</sub> )	(0 <sub>0</sub> , 18 <sub>1</sub> , 29 <sub>2</sub> )	(0 <sub>0</sub> , 21 <sub>1</sub> , 30 <sub>2</sub> )	(0 <sub>0</sub> , 22 <sub>1</sub> , 34 <sub>2</sub> )	(0 <sub>0</sub> , 26 <sub>1</sub> , 14 <sub>2</sub> )
	(0 <sub>0</sub> , 29 <sub>1</sub> , 12 <sub>2</sub> )	(0 <sub>0</sub> , 30 <sub>1</sub> , 1 <sub>2</sub> )	(0 <sub>0</sub> , 31 <sub>1</sub> , 13 <sub>2</sub> )	(0 <sub>0</sub> , 33 <sub>1</sub> , 25 <sub>2</sub> )		

□

**Lemma 4.3.** *If  $v \equiv 15 \pmod{30}$  and  $v > 15$ , then  $(\frac{v-3}{2}, 1) \in \text{HWP}(v; 3, 5)$ .*

*Proof:* Let  $v = 30u + 15$ ,  $u > 0$ . For  $u \leq 3$ , the conclusion comes from Lemmas 4.1 and 4.2. Applying Construction 2.5 with an  $\text{IHW}(45, 15; 3, 5, 15, 0, 6, 1)$  from Lemma 2.3, a  $\text{CF}(30^u; 3, 5, 15, 0)$  from Theorem 2.4 and an  $\text{HW}(45; 3, 5, 21, 1)$  from Lemma 4.1, we get an  $\text{HW}(v; 3, 5, \frac{v-3}{2}, 1)$  for any  $u \geq 4$ . □

**Lemma 4.4.**  *$(\alpha, \beta) \in \text{HWP}(30; 3, 5)$  if and only if  $\alpha + \beta = 14$ .*

*Proof:* Let the vertex set be  $\Gamma = Z_{10} \times Z_3$  and the 1-factor be  $\text{Cay}(\Gamma, \{5\} \times \{0\})$ . For  $(\alpha, \beta) = (10, 4)$ , we get the conclusion by using Construction 2.6 with an  $\text{HW}(10; 3, 5, 0, 4)$  and an  $\text{HW}(K_3[10]; 3, 5, 10, 0)$  from Theorem 1.1. For all the other cases, the methods of generating the required  $\alpha$   $C_3$ -factors and  $\beta$   $C_5$ -factors are listed in Table 1. For  $C_3$ -factors, here are three methods. (1) From a  $C_3$ -factorization of certain Cayley graphs; (2) From several initial  $C_3$ -factors  $P_i$ s by  $(-, +1 \pmod{3})$  or  $(+2 \pmod{10}, -)$ ; (3) From several cycle sets  $Q_i$ s by  $(+2 \pmod{10}, -)$ , note that each  $Q_i$  will generate a  $C_3$ -factor by  $(+2 \pmod{10}, -)$ , since the two elements having the same subscript in  $Q_i$  have different parity. For  $C_5$ -factors, only the first two methods are applied, and the initial  $C_5$ -factors are denoted by  $P'_i$  in Table 1. For the sake of brevity, we list the cycles of  $P_i$ ,  $P'_i$  and  $Q_i$  in Appendix A. □

**Lemma 4.5.** *For each  $(\alpha, \beta) \in \{(0, 22), (6, 16), (12, 10)\}$ ,  $(\alpha, \beta) \in \text{HWP}(K_4[15]; 3, 5)$ .*

*Proof:* Let the vertex set be  $Z_{15} \times Z_4$ , and the four parts of  $K_4[15]$  be  $Z_{15} \times \{i\}$ ,  $i \in Z_4$ . The  $\alpha$   $C_3$ -factors will be obtained from  $\alpha$  3-cycles from a cycle set  $T$  by  $(+3 \pmod{15}, +1 \pmod{4})$ . Note that the first coordinate of the 3 elements in each cycle from  $T$  are different modulo 3, so each cycle of  $T$  will generate a  $C_3$ -factor by  $(+3 \pmod{15}, +1 \pmod{4})$ .



**Table 1** HWP(30; 3, 5)

$(\alpha, \beta)$	$C_3$ -factor	$C_5$ -factor
(1, 13)	1: $Cay(\Gamma, \{0\} \times \{\pm 1\})$	12: $P'_1, P'_2, P'_3, P'_4 (-, +1 \pmod{3})$ 1: $Cay(\Gamma, \{\pm 2\} \times \{0\})$
(2, 12)	2: $Q_1, Q_2$	10: $P'_1, P'_2 (+2 \pmod{10}, -)$ 1: $Cay(\Gamma, \{\pm 2\} \times \{0\})$ 1: $Cay(\Gamma, \{\pm 4\} \times \{0\})$
(3, 11)	2: $Q_1, Q_2$ 1: $Cay(\Gamma, \{0\} \times \{\pm 1\})$	10: $P'_1, P'_2 (+2 \pmod{10}, -)$ 1: $Cay(\Gamma, \{\pm 2\} \times \{0\})$
(4, 10)	4: $Q_1, Q_2, Q_3, Q_4$	10: $P'_1, P'_2 (+2 \pmod{10}, -)$
(5, 9)	5: Lemma 2.1 with $a = i = 1$	9: $P'_1, P'_2, P'_3 (-, +1 \pmod{3})$
(6, 8)	6: $P_1, P_2 (-, +1 \pmod{3})$	6: $P'_1, P'_2 (-, +1 \pmod{3})$ 1: $Cay(\Gamma, \{\pm 2\} \times \{0\})$ 1: $Cay(\Gamma, \{\pm 4\} \times \{0\})$
(7, 7)	6: $P_1, P_2 (-, +1 \pmod{3})$ 1: $Cay(\Gamma, \{0\} \times \{\pm 1\})$	6: $P'_1, P'_2 (-, +1 \pmod{3})$ 1: $Cay(\Gamma, \{\pm 2\} \times \{0\})$
(8, 6)	5: $P_1 (+2 \pmod{10}, -)$ 3: $Q_1, Q_2, Q_3$	5: $P'_1 (+2 \pmod{10}, -)$ 1: $Cay(\Gamma, \{\pm 2\} \times \{0\})$
(9, 5)	5: $P_1 (+2 \pmod{10}, -)$ 3: $Q_1, Q_2, Q_3$ 1: $Cay(\Gamma, \{0\} \times \{\pm 1\})$	5: $P'_1 (+2 \pmod{10}, -)$
(11, 3)	6: $P_1, P_2 (-, +1 \pmod{3})$ 5: Lemma 2.1 with $a = i = 1$	3: $P'_1 (-, +1 \pmod{3})$
(12, 2)	10: $P_1, P_2 (+2 \pmod{10}, -)$ 2: $Q_1, Q_2$	1: $Cay(\Gamma, \{\pm 2\} \times \{0\})$ 1: $Cay(\Gamma, \{\pm 4\} \times \{0\})$
(13, 1)	10: $P_1, P_2 (+2 \pmod{10}, -)$ 3: $Q_1, Q_2, Q_3$	1: $Cay(\Gamma, \{\pm 2\} \times \{0\})$

For  $\beta$   $C_5$ -factors, ten of them will be obtained from two cycle sets  $Q'_1$  and  $Q'_2$ . Here  $\{Q'_i + (5j + k)_0 \mid j = 0, 1, 2\}$  is a  $C_5$ -factor for any  $i = 1, 2$  and  $k = 0, 1, \dots, 4$  since these 5 elements having the same subscript in  $Q'_i$  are different modulo 5. The other  $\beta - 10$   $C_5$ -factors will be obtained from  $\beta - 10$  5-cycles in a cycle set  $T'$  by  $(+5 \pmod{15}, +1 \pmod{4})$ , since the first coordinate of the 5 elements in each cycle from  $T'$  are different modulo 5. We list the 1-factor  $I$  and the cycles in  $T, Q'_1, Q'_2$  and  $T'$  in Appendix B.  $\square$

**Lemma 4.6.**  $(\alpha, \beta) \in \text{HWP}(60; 3, 5)$  if and only if  $\alpha + \beta = 29$ .

*Proof:* By Theorem 1.3, there is an  $\text{HWP}(15; 3, 5, \alpha_1, 7 - \alpha_1)$  for any  $0 \leq \alpha_1 \leq 7$  and  $\alpha_1 \neq 6$ . Apply Construction 2.6 with an  $\text{HW}(K_4[15]; 3, 5, \alpha_2, 22 - \alpha_2)$  for  $\alpha_2 = 0, 6, 12$  from Lemma 4.5 to get an  $\text{HW}(60; 3, 5, \alpha_1 + \alpha_2, 29 - \alpha_1 - \alpha_2)$ . Thus we have  $(\alpha, \beta) \in \text{HWP}(60; 3, 5)$  for  $0 \leq \alpha \leq 19$  and  $\alpha \neq 18$ .

Similarly, for  $(\alpha, \beta) = (20, 9), (25, 4)$ , an  $\text{HW}(60; 3, 5, \alpha, \beta)$  can be obtained from the existence of an  $\text{HW}(K_3[20]; 3, 5, 20, 0)$ , an  $\text{HW}(20; 3, 5, 0, 9)$ , an  $\text{HW}(K_6[10]; 3, 5, 25, 0)$  and an  $\text{HW}(10; 3, 5, 0, 4)$  from Theorem 1.1.

For all the other cases, let the vertex set be  $\Gamma = Z_{15} \times Z_4$  and the 1-factor be  $Cay(\Gamma, \{0\} \times \{2\})$ , the methods of generating the required  $\alpha$   $C_3$ -factors and  $\beta$   $C_5$ -factors are given in Table 2. For generating  $C_3$ -factors, here are five methods. (1) From a  $C_3$ -factorization of certain Cayley graphs; (2) From an initial  $C_3$ -factor  $P$  by  $(+1 \pmod{15}, -)$ ; (3) From several cycle sets  $Q_i$ s, note that  $\{Q_i + (3j + k)_0 \mid j = 0, 1, \dots, 4\}$  is a  $C_3$ -factor for  $k = 0, 1, 2$  since

these 3 elements having the same subscript in  $Q_i$  are different modulo 3; (4) From a cycle in  $T$  by  $(+1 \pmod{15}, +1 \pmod{4})$ . A  $C_3$ -factor  $F$  can be obtained from the cycle in  $T$  by  $(+3 \pmod{15}, +1 \pmod{4})$ . Then three  $C_3$ -factors can be generated from  $F$  by  $(+i \pmod{15}, -)$ ,  $i = 0, 1, 2$ ; (5) From a cycle set  $S$  by  $(+3 \pmod{15}, +1 \pmod{4})$ . Note that the first coordinate of the 3 elements in each cycle from  $S$  are different modulo 3, so each cycle of  $S$  will generate a  $C_3$ -factor by  $(+3 \pmod{15}, +1 \pmod{4})$ .

For  $C_5$ -factors, three methods are applied. (1) From a  $C_5$ -factorization of certain Cayley graphs; (2) From a cycle set  $Q'$ , where  $\{Q' + (5j + k)_0 \mid j = 0, 1, 2\}$  is a  $C_5$ -factor for  $k = 0, 1, \dots, 4$  since these 5 elements having the same subscript in  $Q'$  are different modulo 5; (3) From a cycle set  $T'$  by  $(+5 \pmod{15}, +1 \pmod{4})$ . The cycles of  $P, Q_i, Q', S, T$  and  $T'$  are given in Appendix C.  $\square$

**Table 2** HWP(60; 3, 5)

$(\alpha, \beta)$	$C_3$ -factor	$C_5$ -factor
(18, 11)	15: $P$ 3: $Q_1$	5: $Q'$ 5: $T'$ 1: $\text{Cay}(\Gamma, \{\pm 6\} \times \{0\})$
(21, 8)	15: $P$ 3: $Q_1$ 3: $T$	5: $Q'$ 3: $T'$
(22, 7)	15: $P$ 3: $Q_1$ 3: $T$ 1: $\text{Cay}(\Gamma, \{\pm 5\} \times \{0\})$	5: $Q'$ 1: $\text{Cay}(\Gamma, \{\pm 3\} \times \{0\})$ 1: $\text{Cay}(\Gamma, \{\pm 6\} \times \{0\})$
(23, 6)	15: $P$ 3: $Q_1$ 5: $S$	5: $Q'$ 1: $\text{Cay}(\Gamma, \{\pm 6\} \times \{0\})$
(24, 5)	15: $P$ 9: $Q_i, 1 \leq i \leq 3$	3: $T'$ 1: $\text{Cay}(\Gamma, \{\pm 3\} \times \{0\})$ 1: $\text{Cay}(\Gamma, \{\pm 6\} \times \{0\})$
(26, 3)	15: $P$ 6: $Q_1, Q_2$ 5: $S$	3: $T'$
(27, 2)	15: $P$ 12: $Q_i, 1 \leq i \leq 4$	1: $\text{Cay}(\Gamma, \{\pm 3\} \times \{0\})$ 1: $\text{Cay}(\Gamma, \{\pm 6\} \times \{0\})$
(28, 1)	15: $P$ 12: $Q_i, 1 \leq i \leq 4$ 1: $\text{Cay}(\Gamma, \{\pm 5\} \times \{0\})$	1: $\text{Cay}(\Gamma, \{\pm 6\} \times \{0\})$

**Lemma 4.7.** *If  $v \equiv 0 \pmod{30}$ , then  $(\alpha, \beta) \in \text{HWP}(v; 3, 5)$  for any  $\alpha + \beta = \frac{v-2}{2}$ .*

*Proof:* Let  $v = 30u$ ,  $u \geq 1$ . For  $u \leq 2$ , the conclusion comes from Lemmas 4.4 and 4.6. For  $u = 3$ , start with an  $\text{HW}(K_3[3]; 3, 5, 3, 0)$ , an  $\text{HW}(C_3[10]; 3, 5, 10, 0)$  and an  $\text{HW}(C_3[10]; 3, 5, 0, 10)$  from Theorem 1.1, apply Construction 2.7 with  $s = 10$  and  $t_i \in \{0, 10\}$  to get an  $\text{HW}(K_3[30]; 3, 5, \sum_{i=1}^3 t_i, 30 - \sum_{i=1}^3 t_i)$ . Then we apply Construction 2.6 with an  $\text{HW}(30; 3, 5, \alpha', 14 - \alpha')$ ,  $0 \leq \alpha' \leq 14$ , from Lemma 4.4 to obtain an  $\text{HW}(90; 3, 5, \sum_{i=1}^3 t_i + \alpha', 30 - \sum_{i=1}^3 t_i + (14 - \alpha'))$ . Thus we have obtained an  $\text{HW}(90; 3, 5, \alpha, \beta)$  for any  $\alpha + \beta = 44$  since  $\sum_{i=1}^3 t_i + \alpha'$  can cover the integers from 0 to 44.

For  $u \geq 4$ , similarly, we start with an  $\text{HW}(K_u[6]; 3, 5, 3u - 3, 0)$ , an  $\text{HW}(C_3[5]; 3, 5, 5, 0)$  and an  $\text{HW}(C_3[5]; 3, 5, 0, 5)$  from Theorem 1.1, and apply Construction 2.7 with  $s = 5$  and  $t_i \in \{0, 5\}$  to get an  $\text{HW}(K_u[30]; 3, 5, \sum_{i=1}^{3u-3} t_i, 15u - 15 - \sum_{i=1}^{3u-3} t_i)$ . Further, applying Construction 2.6 with an  $\text{HW}(30; 3, 5, \alpha', 14 - \alpha')$ ,  $0 \leq \alpha' \leq 14$ , from Lemma 4.4, we can obtain an  $\text{HW}(30u; 3, 5, \alpha' + \sum_{i=1}^{3u-3} t_i, 14 - \alpha' + 15u - 15 - \sum_{i=1}^{3u-3} t_i)$ . It's easy to prove that  $\alpha' + \sum_{i=1}^{3u-3} t_i$  can cover the integers from 0 to  $15u - 1$ . The proof is complete.  $\square$

Combining Theorem 1.3, Lemmas 4.3 and 4.7, we have the following theorem.

**Theorem 4.8.**  $(\alpha, \beta) \in \text{HWP}(v; 3, 5)$  if and only if  $v \equiv 0 \pmod{15}$ ,  $\alpha + \beta = \lfloor \frac{v-1}{2} \rfloor$  and  $(\alpha, \beta, v) \neq (6, 1, 15)$ .

## 5 $\text{HWP}(v; 3, 7)$

In this section, we shall solve the three left cases in [22] for the existence of an  $\text{HW}(v; 3, 7, \alpha, \beta)$  when  $v \equiv 21 \pmod{42}$ . Then we continue to consider the case  $v \equiv 0 \pmod{42}$ .

**Lemma 5.1.** For each  $(\alpha, \beta) \in \{(2, 8), (4, 6), (6, 4)\}$ ,  $(\alpha, \beta) \in \text{HWP}(21; 3, 7)$ .

*Proof:* Let the vertex set be  $\Gamma = Z_7 \times Z_3$ . For  $\alpha = 2$ , the required two  $C_3$ -factors can be generated from two base cycles  $(0_0, 1_1, 2_2)$  and  $(0_0, 4_1, 1_2)$  by  $(+1 \pmod{7}, -)$ . Seven of the required  $C_7$ -factors can be obtained from an initial  $C_7$ -factor  $P = \{(0_0, 5_2, 6_0, 1_1, 3_0, 3_1, 3_2), (2_2, 5_0, 4_1, 6_2, 4_2, 5_1, 6_1), (0_1, 2_1, 0_2, 1_2, 1_0, 2_0, 4_0)\}$  by  $(+1 \pmod{7}, -)$ . The last  $C_7$ -factor is  $\text{Cay}(\Gamma, \{\pm 3\} \times \{0\})$ .

For  $\alpha = 4$ , a  $C_3$ -factor is  $\text{Cay}(\Gamma, \{0\} \times \{\pm 1\})$ , and the other 3  $C_3$ -factors can be generated from an initial  $C_3$ -factor  $P$  by  $(-, +1 \pmod{3})$ . All  $C_7$ -factors can be generated from two initial  $C_7$ -factors  $Q_1$  and  $Q_2$  by  $(-, +1 \pmod{3})$ .  $P$ ,  $Q_1$  and  $Q_2$  are listed below.

$P$	$(0_0, 1_1, 2_2)$	$(3_0, 4_1, 5_2)$	$(6_0, 0_1, 3_1)$	$(1_2, 4_2, 6_1)$	$(2_0, 0_2, 4_0)$	$(5_0, 1_0, 3_2)$	$(2_1, 5_1, 6_2)$
$Q_1$	$(0_0, 5_2, 1_1, 0_1, 3_0, 2_2, 1_2)$	$(4_1, 2_0, 6_0, 5_0, 0_2, 6_2, 3_2)$	$(3_1, 2_1, 4_0, 6_1, 1_0, 4_2, 5_1)$				
$Q_2$	$(0_0, 2_0, 1_1, 4_2, 6_2, 0_1, 4_0)$	$(2_2, 5_0, 2_1, 3_0, 0_2, 5_2, 6_1)$	$(4_1, 3_1, 1_0, 6_0, 3_2, 1_2, 5_1)$				

For  $\alpha = 6$ , All  $C_3$ -factors can be generated from two initial  $C_3$ -factors  $P_1$  and  $P_2$  by  $(-, +1 \pmod{3})$ . Three  $C_7$ -factors can be generated from an initial  $C_7$ -factor  $Q$  by  $(-, +1 \pmod{3})$ . The last  $C_7$ -factor is  $\text{Cay}(\Gamma, \{\pm 3\} \times \{0\})$ .  $P_1$ ,  $P_2$  and  $Q$  are listed below.

$P_1$	$(0_0, 1_1, 2_2)$	$(3_0, 4_1, 5_2)$	$(6_0, 0_1, 4_2)$	$(1_2, 3_1, 0_2)$	$(2_0, 6_1, 3_2)$	$(5_0, 2_1, 4_0)$	$(1_0, 5_1, 6_2)$
$P_2$	$(0_0, 5_2, 0_1)$	$(1_1, 1_2, 2_1)$	$(2_2, 3_0, 3_2)$	$(4_1, 3_1, 4_0)$	$(6_0, 5_0, 5_1)$	$(2_0, 0_2, 6_2)$	$(4_2, 6_1, 1_0)$
$Q$	$(0_0, 1_2, 6_2, 6_1, 4_1, 2_1, 2_0)$	$(1_1, 4_2, 5_1, 0_1, 4_0, 2_2, 5_0)$	$(3_0, 0_2, 6_0, 3_2, 5_2, 3_1, 1_0)$				

$\square$

**Lemma 5.2.**  $(\alpha, \beta) \in \text{HWP}(42; 3, 7)$  if and only if  $\alpha + \beta = 20$ .

*Proof:* For  $(\alpha, \beta) = (14, 6)$ , we get the conclusion by using Construction 2.6 with an  $\text{HW}(14; 3, 7, 0, 6)$  and an  $\text{HW}(K_3[14]; 3, 7, 14, 0)$  from Theorem 1.1.

For all the other cases, let the vertex set be  $\Gamma = Z_{14} \times Z_3$  and the 1-factor be  $\text{Cay}(\Gamma, \{7\} \times \{0\})$ . The methods of generating the required  $\alpha$   $C_3$ -factors and  $\beta$   $C_7$ -factors are listed in the following table.

**Table 3** HWP(42; 3, 7)

$(\alpha, \beta)$	$C_3$ -factor	$C_7$ -factor
(1, 19)	1: $\text{Cay}(\Gamma, \{0\} \times \{\pm 1\})$	18: $P'_1, P'_2, P'_3 (+7 \pmod{14}, +1 \pmod{3})$ 1: $\text{Cay}(\Gamma, \{\pm 4\} \times \{0\})$
(2, 18)	2: $Q_1, Q_2$	14: $P'_1, P'_2 (+2 \pmod{14}, -)$ 4: $T'$
(3, 17)	3: $Q_i, 1 \leq i \leq 3$	14: $P'_1, P'_2 (+2 \pmod{14}, -)$ 1: $\text{Cay}(\Gamma, \{\pm 2\} \times \{0\})$ 1: $\text{Cay}(\Gamma, \{\pm 4\} \times \{0\})$ 1: $\text{Cay}(\Gamma, \{\pm 6\} \times \{0\})$
(4, 16)	4: $Q_i, 1 \leq i \leq 4$	14: $P'_1, P'_2 (+2 \pmod{14}, -)$ 1: $\text{Cay}(\Gamma, \{\pm 2\} \times \{0\})$ 1: $\text{Cay}(\Gamma, \{\pm 4\} \times \{0\})$
(5, 15)	5: Lemma 2.1 with $a = 2$ and $i = 1$	12: $P'_1, P'_2 (+7 \pmod{14}, +1 \pmod{3})$ 1: $\text{Cay}(\Gamma, \{\pm 2\} \times \{0\})$ 1: $\text{Cay}(\Gamma, \{\pm 4\} \times \{0\})$ 1: $\text{Cay}(\Gamma, \{\pm 6\} \times \{0\})$
(6, 14)	6: $P_1 (+7 \pmod{14}, +1 \pmod{3})$	12: $P'_1, P'_2 (+7 \pmod{14}, +1 \pmod{3})$ 1: $\text{Cay}(\Gamma, \{\pm 2\} \times \{0\})$ 1: $\text{Cay}(\Gamma, \{\pm 4\} \times \{0\})$
(7, 13)	6: $P_1 (+7 \pmod{14}, +1 \pmod{3})$ 1: $\text{Cay}(\Gamma, \{0\} \times \{\pm 1\})$	12: $P'_1, P'_2 (+7 \pmod{14}, +1 \pmod{3})$ 1: $\text{Cay}(\Gamma, \{\pm 4\} \times \{0\})$
(8, 12)	5: Lemma 2.1 with $a = 11$ and $i = 1$ 3: $T$	12: $P'_1, P'_2 (+7 \pmod{14}, +1 \pmod{3})$
(9, 11)	9: $T$	6: $P'_1 (+7 \pmod{14}, +1 \pmod{3})$ 3: $Q'$ 1: $\text{Cay}(\Gamma, \{\pm 2\} \times \{0\})$ 1: $\text{Cay}(\Gamma, \{\pm 4\} \times \{0\})$
(10, 10)	5: Lemma 2.1 with $a = 11$ and $i = 1$ 5: $Q_i, 1 \leq i \leq 5$	7: $P'_1 (+2 \pmod{14}, -)$ 1: $\text{Cay}(\Gamma, \{\pm 2\} \times \{0\})$ 1: $\text{Cay}(\Gamma, \{\pm 4\} \times \{0\})$ 1: $\text{Cay}(\Gamma, \{\pm 6\} \times \{0\})$
(11, 9)	5: Lemma 2.1 with $a = 11$ and $i = 1$ 6: $P_1 (+7 \pmod{14}, +1 \pmod{3})$	6: $P'_1 (+7 \pmod{14}, +1 \pmod{3})$ 1: $\text{Cay}(\Gamma, \{\pm 2\} \times \{0\})$ 1: $\text{Cay}(\Gamma, \{\pm 4\} \times \{0\})$ 1: $\text{Cay}(\Gamma, \{\pm 6\} \times \{0\})$
(12, 8)	12: $P_1, P_2 (+7 \pmod{14}, +1 \pmod{3})$	6: $P'_1 (+7 \pmod{14}, +1 \pmod{3})$ 1: $\text{Cay}(\Gamma, \{\pm 2\} \times \{0\})$ 1: $\text{Cay}(\Gamma, \{\pm 4\} \times \{0\})$
(13, 7)	12: $P_1, P_2 (+7 \pmod{14}, +1 \pmod{3})$ 1: $\text{Cay}(\Gamma, \{0\} \times \{\pm 1\})$	6: $P'_1 (+7 \pmod{14}, +1 \pmod{3})$ 1: $\text{Cay}(\Gamma, \{\pm 4\} \times \{0\})$
(15, 5)	6: $P_1 (+7 \pmod{14}, +1 \pmod{3})$ 9: $T$	3: $Q'$ 1: $\text{Cay}(\Gamma, \{\pm 2\} \times \{0\})$ 1: $\text{Cay}(\Gamma, \{\pm 4\} \times \{0\})$
(16, 4)	6: $P_1 (+7 \pmod{14}, +1 \pmod{3})$ 9: $T$ 1: $\text{Cay}(\Gamma, \{0\} \times \{\pm 1\})$	3: $Q'$ 1: $\text{Cay}(\Gamma, \{\pm 4\} \times \{0\})$
(17, 3)	14: $P_1 (+1 \pmod{14}, -)$ 2: $S$ 1: $\text{Cay}(\Gamma, \{0\} \times \{\pm 1\})$	1: $\text{Cay}(\Gamma, \{\pm 2\} \times \{0\})$ 1: $\text{Cay}(\Gamma, \{\pm 4\} \times \{0\})$ 1: $\text{Cay}(\Gamma, \{\pm 6\} \times \{0\})$
(18, 2)	14: $P_1 (+1 \pmod{14}, -)$ 3: $S$ 1: $\text{Cay}(\Gamma, \{0\} \times \{\pm 1\})$	1: $\text{Cay}(\Gamma, \{\pm 2\} \times \{0\})$ 1: $\text{Cay}(\Gamma, \{\pm 4\} \times \{0\})$
(19, 1)	14: $P_1 (+1 \pmod{14}, -)$ 5: Lemma 2.1 with $a = 1$ and $i = 1$	1: $\text{Cay}(\Gamma, \{\pm 4\} \times \{0\})$

Here are five methods to get  $C_3$ -factors. (1) From  $C_3$ -factorization of certain Cayley graphs; (2) From several initial  $C_3$ -factors  $P_i$ s by  $(+7 \pmod{14}, +1 \pmod{3})$  or  $(+1 \pmod{14}, -)$ ; (3) From several cycle sets  $Q_i$ s each of which can generate a  $C_3$ -factor by  $(+2 \pmod{14}, -)$  since the two elements having the same subscript in  $Q_i$  have different parity; (4) From a cycle set  $T$ , each cycle of which can generate a  $C_3$ -factor  $F$  by  $(+1 \pmod{14}, -)$ , then 3  $C_3$ -factors can be generated from  $F$  by  $(-, +1 \pmod{3})$ ; (5) From a partial  $C_3$ -factor  $S$ , each cycle of  $S$  will generate a  $C_3$ -factor by  $(+1 \pmod{14}, -)$ .

For  $C_7$ -factors, we have the following four methods. (1) From  $C_7$ -factorization of certain Cayley graphs; (2) From several initial  $C_7$ -factors  $P'_i$ s by  $(+7 \pmod{14}, +1 \pmod{3})$  or  $(+2 \pmod{14}, -)$ ; (3) From a cycle set  $Q'$ , since these 7 elements having the same subscript in  $Q'$  are different modulo 7, so  $Q'$  can generate a  $C_7$ -factor  $F$  by  $(+7 \pmod{14}, -)$ , then 3  $C_7$ -factors can be generated from  $F$  by  $(-, +1 \pmod{3})$ ; (4) From a cycle set  $T'$ , each cycle of which will generate a  $C_7$ -factor by  $(+7 \pmod{14}, +1 \pmod{3})$ .

The cycles of  $P_i$ ,  $P'_i$ ,  $Q_i$ ,  $Q'$ ,  $T$ ,  $T'$  and  $S$  are given in Appendix D.  $\square$

**Lemma 5.3.** *For any  $(\alpha, \beta) \in \{(0, 31), (9, 22), (18, 13), (24, 7)\}$ ,  $(\alpha, \beta) \in \text{HWP}(K_4[21]; 3, 7)$ .*

*Proof:* Let the vertex set be  $Z_{21} \times Z_4$ , and the four parts of  $K_4[21]$  be  $Z_{21} \times \{i\}$ ,  $i \in Z_4$ . For any  $\alpha > 0$ , the required  $\alpha$   $C_3$ -factors are

$$F_i^k = \{Q_i + (3j + k)_0 \mid j = 0, 1, \dots, 6\}, \quad 1 \leq i \leq \alpha/3, \quad k = 0, 1, 2.$$

For  $C_7$ -factors, some of them will be obtained from several cycle sets  $Q'_i$ . Here  $\{Q'_i + (7j + k)_0 \mid j = 0, 1, 2\}$  is a  $C_7$ -factor for any  $k = 0, 1, \dots, 6$  since these 7 elements having the same subscript in  $Q'_i$  are different modulo 7. The other  $C_7$ -factors will be obtained from a cycle set  $T'$  by  $(+7 \pmod{21}, +1 \pmod{4})$ , since the first coordinate of the 7 elements in each cycle from  $T'$  are different modulo 7. For the sake of brevity, we list the 1-factor  $I$  and the cycles of  $Q_i$ ,  $Q'_i$  and  $T'$  in Appendix E.  $\square$

**Lemma 5.4.**  *$(\alpha, \beta) \in \text{HWP}(84; 3, 7)$  if and only if  $\alpha + \beta = 41$ .*

*Proof:* Applying Construction 2.6 with an  $\text{HW}(K_4[21]; 3, 7, \alpha_1, 31 - \alpha_1)$ ,  $\alpha_1 = 0, 9, 18, 24$ , from Lemma 5.3 and an  $\text{HW}(21; 3, 7, \alpha_2, 10 - \alpha_2)$ ,  $0 \leq \alpha_2 \leq 10$ , from Theorem 1.4 and Lemma 5.1, we can get an  $\text{HW}(84; 3, 7, \alpha_1 + \alpha_2, 41 - \alpha_1 - \alpha_2)$ . Thus,  $(\alpha, \beta) \in \text{HWP}(84; 3, 7)$  for  $0 \leq \alpha \leq 34$ . Similarly, for  $(\alpha, \beta) = (35, 6)$ , we can obtain the conclusion with an  $\text{HW}(14; 3, 7, 0, 6)$  and an  $\text{HW}(K_6[14]; 3, 7, 35, 0)$  from Theorem 1.1. For all the other cases, let the vertex set be  $Z_{21} \times Z_4$ . The methods of generating the required  $\alpha$   $C_3$ -factors and  $\beta$   $C_7$ -factors are listed in Table 4.

For generating  $C_3$ -factors, here are four methods. (1) From a  $C_3$ -factorization of certain Cayley graphs; (2) From an initial  $C_3$ -factor  $P$  by  $(+1 \pmod{21}, -)$ ; (3) From several cycle

sets  $Q_i$ s, note that  $\{Q_i + (3j + k)_0 | j = 0, 1, \dots, 6\}$  is a  $C_3$ -factor for any  $k = 0, 1, 2$  since these 3 elements having the same subscript in  $Q_i$  are different modulo 3; (4) From a cycle set  $T$  by  $(+3 \pmod{21}, +1 \pmod{4})$ , where the first coordinate of the 3 elements in each cycle from  $T$  are different modulo 3, so each cycle of  $T$  will generate a  $C_3$ -factor by  $(+3 \pmod{21}, +1 \pmod{4})$ . The required  $C_7$ -factors are given from a  $C_7$ -factorization of certain Cayley graphs or from a cycle set  $T'$  by  $(+7 \pmod{21}, +1 \pmod{4})$ . The cycles of  $P$ ,  $Q_i$ ,  $T$  and  $T'$  are given in Appendix F.  $\square$

**Table 4** HWP(84; 3, 7)

$(\alpha, \beta)$	$C_3$ -factor	$C_7$ -factor	1-factor
(36, 5)	21: $P$ 15: $Q_i, 1 \leq i \leq 5$	4: $T'$ 1: $\text{Cay}(\Gamma, \{\pm 9\} \times \{0\})$	$\{(i_0, (i + 17)_1), (i_2, (i + 4)_3)   i \in \mathbb{Z}_{21}\}$
(37, 4)	21: $P$ 15: $Q_i, 1 \leq i \leq 5$ 1: $\text{Cay}(\Gamma, \{\pm 7\} \times \{0\})$	4: $T'$	$\{(i_0, (i + 16)_1), (i_2, (i + 12)_3)   i \in \mathbb{Z}_{21}\}$
(38, 3)	21: $P$ 12: $Q_i, 1 \leq i \leq 4$ 5: $T$	1: $\text{Cay}(\Gamma, \{\pm 3\} \times \{0\})$ 1: $\text{Cay}(\Gamma, \{\pm 6\} \times \{0\})$ 1: $\text{Cay}(\Gamma, \{\pm 9\} \times \{0\})$	$\{(i_0, (i + 1)_1), (i_2, (i + 13)_3)   i \in \mathbb{Z}_{21}\}$
(39, 2)	21: $P$ 18: $Q_i, 1 \leq i \leq 6$	1: $\text{Cay}(\Gamma, \{\pm 6\} \times \{0\})$ 1: $\text{Cay}(\Gamma, \{\pm 9\} \times \{0\})$	$\{(i_0, (i + 20)_1), (i_2, i_3)   i \in \mathbb{Z}_{21}\}$
(40, 1)	21: $P$ 18: $Q_i, 1 \leq i \leq 6$ 1: $\text{Cay}(\Gamma, \{\pm 7\} \times \{0\})$	1: $\text{Cay}(\Gamma, \{\pm 9\} \times \{0\})$	$\{(i_0, (i + 5)_1), (i_2, (i + 11)_3)   i \in \mathbb{Z}_{21}\}$

**Lemma 5.5.** *If  $v \equiv 0 \pmod{42}$ , then  $(\alpha, \beta) \in \text{HWP}(v; 3, 7)$  with  $\alpha + \beta = \frac{v-2}{2}$ .*

*Proof:* Let  $v = 42u$ ,  $u \geq 1$ . For  $u \leq 2$ , the conclusion comes from Lemmas 5.2 and 5.4.

For  $u = 3$ , start with an  $\text{HW}(K_3[3]; 3, 7, 3, 0)$ , an  $\text{HW}(C_3[14]; 3, 7, 14, 0)$  and an  $\text{HW}(C_3[14]; 3, 7, 0, 14)$  from Theorem 1.1, apply Construction 2.7 with  $s = 14$  and  $t_i \in \{0, 14\}$  to get an  $\text{HW}(K_3[42]; 3, 7, \sum_{i=1}^3 t_i, 42 - \sum_{i=1}^3 t_i)$ . Further, applying Construction 2.6 with an  $\text{HW}(42; 3, 7, \alpha', 20 - \alpha')$ ,  $0 \leq \alpha' \leq 20$ , from Lemma 5.2 to obtain an  $\text{HW}(126; 3, 7, \sum_{i=1}^3 t_i + \alpha', 42 - \sum_{i=1}^3 t_i + (20 - \alpha'))$ . Thus we have obtained an  $\text{HW}(126; 3, 7, \alpha, \beta)$  for any  $\alpha + \beta = 62$  since  $\sum_{i=1}^3 t_i + \alpha'$  can cover the integers from 0 to 62.

For  $u \geq 4$ , similarly, start with an  $\text{HW}(K_u[6]; 3, 7, 3u - 3, 0)$ , an  $\text{HW}(C_3[7]; 3, 7, 7, 0)$  and an  $\text{HW}(C_3[7]; 3, 7, 0, 7)$  from Theorem 1.1, and apply Construction 2.7 with  $s = 7$  and  $t_i \in \{0, 7\}$  to get an  $\text{HW}(K_u[42]; 3, 7, \sum_{i=1}^{3u-3} t_i, 21u - 21 - \sum_{i=1}^{3u-3} t_i)$ . Further, applying Construction 2.6 with an  $\text{HW}(42; 3, 7, \alpha', 20 - \alpha')$ ,  $0 \leq \alpha' \leq 20$ , from Lemma 5.2, we can obtain an  $\text{HW}(42u; 3, 7, \alpha' + \sum_{i=1}^{3u-3} t_i, 20 - \alpha' + 21u - 21 - \sum_{i=1}^{3u-3} t_i)$ . It's easy to prove that  $\alpha' + \sum_{i=1}^{3u-3} t_i$  can cover the integers from 0 to  $21u - 1$ . The proof is complete.  $\square$

Combining Theorem 1.4, Lemmas 5.1 and 5.5, we have the following theorem.

**Theorem 5.6.**  $(\alpha, \beta) \in \text{HWP}(v; 3, 7)$  if and only if  $v \equiv 0 \pmod{21}$  and  $\alpha + \beta = \lfloor \frac{v-1}{2} \rfloor$ .

Combining Theorems 3.4, 4.8 and 5.6, we have proved Theorem 1.5.

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## Appendix A for Lemma 4.4

$(\alpha, \beta) = (1, 13) :$

$P'_1$	$(0_1, 2_0, 4_2, 1_2, 3_1)$ $(0_0, 1_1, 2_2, 3_0, 5_1)$	$(5_2, 6_0, 7_1, 8_2, 9_0)$ $(4_1, 0_2, 6_2, 1_0, 7_0)$	$(5_0, 7_2, 9_1, 6_1, 8_0)$ $(2_1, 8_1, 3_2, 4_0, 9_2)$
$P'_2$	$(0_0, 4_1, 5_2, 1_1, 8_2)$ $(2_0, 5_0, 1_0, 4_2, 8_1)$	$(2_2, 6_0, 3_0, 7_1, 0_1)$ $(6_1, 3_2, 5_1, 8_0, 9_2)$	$(9_0, 1_2, 9_1, 3_1, 0_2)$ $(7_2, 2_1, 7_0, 4_0, 6_2)$
$P'_3$	$(0_0, 5_2, 2_0, 1_1, 7_2)$ $(6_0, 2_1, 9_1, 4_0, 8_1)$	$(2_2, 1_2, 3_0, 4_2, 3_2)$ $(7_1, 8_0, 1_0, 0_1, 6_2)$	$(4_1, 5_0, 9_2, 8_2, 5_1)$ $(9_0, 3_1, 7_0, 0_2, 6_1)$
$P'_4$	$(0_0, 5_2, 2_0, 1_1, 7_2)$ $(6_0, 2_1, 9_1, 4_0, 8_1)$	$(2_2, 1_2, 3_0, 4_2, 3_2)$ $(7_1, 8_0, 1_0, 0_1, 6_2)$	$(4_1, 5_0, 9_2, 8_2, 5_1)$ $(9_0, 3_1, 7_0, 0_2, 6_1)$

$(\alpha, \beta) = (2, 12) :$

$Q_1$	$(0_0, 1_1, 2_2)$	$(3_0, 4_1, 5_2)$	$Q_2$	$(0_0, 3_0, 7_1)$	$(2_2, 4_1, 1_2)$
$P'_1$	$(0_0, 4_1, 1_1, 3_0, 8_2)$ $(4_2, 5_0, 8_1, 0_2, 4_0)$	$(2_2, 5_2, 6_0, 1_2, 7_1)$ $(6_1, 1_0, 9_2, 9_1, 7_0)$		$(9_0, 2_0, 0_1, 3_1, 8_0)$ $(7_2, 2_1, 6_2, 3_2, 5_1)$	
$P'_2$	$(0_0, 9_0, 5_2, 4_0, 7_2)$ $(3_0, 3_1, 9_2, 0_1, 3_2)$	$(1_1, 8_2, 7_0, 1_2, 0_2)$ $(4_1, 2_0, 6_2, 6_1, 5_1)$		$(2_2, 5_0, 2_1, 6_0, 9_1)$ $(7_1, 8_0, 8_1, 4_2, 1_0)$	

$(\alpha, \beta) = (3, 11) :$

$Q_1$	$(0_0, 1_1, 2_2)$	$(3_0, 4_1, 5_2)$	$Q_2$	$(0_0, 3_0, 7_1)$	$(2_2, 4_1, 1_2)$
$P'_1$	$(0_0, 4_1, 1_1, 3_0, 6_0)$ $(3_1, 8_0, 6_2, 7_2, 0_2)$	$(2_2, 5_2, 7_1, 1_2, 5_0)$ $(6_1, 2_1, 5_1, 9_1, 7_0)$		$(8_2, 9_0, 4_2, 0_1, 2_0)$ $(1_0, 8_1, 3_2, 4_0, 9_2)$	
$P'_2$	$(0_0, 9_0, 2_2, 8_2, 2_1)$ $(4_1, 3_2, 7_2, 6_0, 5_1)$	$(1_1, 5_0, 4_0, 1_2, 8_0)$ $(5_2, 3_1, 9_2, 6_1, 1_0)$		$(3_0, 4_2, 9_1, 0_1, 7_0)$ $(7_1, 0_2, 8_1, 2_0, 6_2)$	

$(\alpha, \beta) = (4, 10) :$

$Q_1$	$(0_0, 1_1, 2_2)$	$(3_0, 4_1, 5_2)$	$Q_2$	$(0_0, 3_0, 7_1)$	$(2_2, 4_1, 1_2)$
$Q_3$	$(0_0, 4_1, 8_2)$	$(1_1, 3_0, 1_2)$	$Q_4$	$(0_0, 5_2, 9_0)$	$(1_1, 4_1, 4_2)$
$P'_1$	$(0_0, 6_0, 1_1, 5_2, 0_1)$ $(4_2, 8_0, 9_2, 3_2, 6_2)$	$(2_2, 3_0, 8_2, 1_2, 2_0)$ $(5_0, 1_0, 4_0, 2_1, 8_1)$		$(4_1, 7_1, 3_1, 0_2, 9_0)$ $(6_1, 5_1, 7_2, 9_1, 7_0)$	
$P'_2$	$(0_0, 2_0, 1_1, 0_2, 4_2)$ $(5_2, 7_2, 4_0, 7_1, 1_0)$	$(2_2, 5_0, 7_0, 6_0, 3_2)$ $(8_2, 3_1, 5_1, 1_2, 2_1)$		$(3_0, 0_1, 8_0, 4_1, 6_2)$ $(9_0, 9_1, 8_1, 6_1, 9_2)$	

$(\alpha, \beta) = (5, 9) :$

$P'_1$	$(0_0, 3_0, 6_0, 1_1, 4_1)$ $(3_1, 7_2, 0_2, 4_2, 6_2)$	$(2_2, 5_2, 8_2, 1_2, 7_1)$ $(8_0, 2_1, 8_1, 9_1, 3_2)$	$(9_0, 2_0, 6_1, 0_1, 5_0)$ $(1_0, 5_1, 9_2, 4_0, 7_0)$
$P'_2$	$(0_0, 7_1, 2_0, 3_0, 9_0)$ $(4_1, 1_2, 5_1, 6_1, 1_0)$	$(1_1, 8_2, 5_0, 0_2, 3_1)$ $(5_2, 2_1, 9_2, 6_0, 8_0)$	$(2_2, 9_1, 4_0, 7_2, 6_2)$ $(0_1, 3_2, 8_1, 4_2, 7_0)$
$P'_3$	$(0_0, 2_0, 5_2, 3_2, 4_2)$ $(4_1, 2_1, 6_0, 9_0, 5_1)$	$(1_1, 0_1, 6_2, 3_0, 9_1)$ $(7_1, 3_1, 7_0, 5_0, 1_0)$	$(2_2, 1_2, 8_1, 4_0, 8_0)$ $(8_2, 7_2, 9_2, 6_1, 0_2)$

$(\alpha, \beta) = (6, 8) :$

$P_1$	$(0_2, 4_0, 7_0)$ $(2_0, 5_0, 7_2)$	$(3_0, 4_1, 5_2)$ $(8_0, 9_1, 1_0)$	$(6_0, 7_1, 8_2)$ $(2_1, 6_2, 9_2)$	$(9_0, 0_1, 3_1)$ $(0_0, 1_1, 3_2)$	$(1_2, 4_2, 6_1)$ $(2_2, 5_1, 8_1)$
$P_2$	$(0_0, 2_2, 0_1)$ $(4_2, 1_0, 5_1)$	$(1_1, 3_0, 8_2)$ $(5_0, 9_1, 8_1)$	$(4_1, 9_0, 2_0)$ $(6_1, 7_0, 9_2)$	$(5_2, 6_0, 3_1)$ $(7_2, 2_1, 4_0)$	$(7_1, 1_2, 0_2)$ $(8_0, 3_2, 6_2)$
$P'_1$	$(0_0, 5_2, 0_1, 6_0, 1_2)$ $(8_2, 9_1, 6_2, 6_1, 4_0)$	$(1_1, 2_2, 3_0, 7_1, 5_0)$ $(9_0, 4_2, 3_2, 3_1, 2_1)$		$(4_1, 7_2, 7_0, 8_0, 5_1)$ $(2_0, 1_0, 9_2, 0_2, 8_1)$	
$P'_2$	$(0_0, 3_1, 6_2, 7_2, 8_1)$ $(3_0, 9_1, 7_0, 1_2, 2_1)$	$(1_1, 4_2, 8_0, 8_2, 1_0)$ $(4_1, 3_2, 7_1, 0_1, 4_0)$		$(2_2, 2_0, 5_1, 5_2, 6_1)$ $(6_0, 5_0, 9_2, 9_0, 0_2)$	

$(\alpha, \beta) = (7, 7) :$

$P_1$	(02, 40, 70)	(30, 41, 52)	(60, 71, 82)	(90, 01, 31)	(12, 42, 61)
	(20, 50, 72)	(80, 91, 10)	(00, 11, 22)	(21, 32, 81)	(51, 62, 92)
$P_2$	(00, 52, 82)	(11, 30, 71)	(22, 41, 90)	(60, 01, 91)	(12, 80, 02)
	(20, 32, 92)	(31, 61, 70)	(42, 10, 51)	(50, 21, 62)	(72, 40, 81)
$P'_1$	(00, 60, 12, 22, 61)	(11, 90, 20, 30, 51)	(41, 01, 40, 80, 32)		
	(52, 62, 91, 70, 92)	(71, 31, 02, 72, 10)	(82, 50, 81, 42, 21)		
$P'_2$	(00, 12, 40, 30, 91)	(11, 20, 02, 71, 50)	(22, 70, 42, 60, 81)		
	(41, 51, 90, 80, 92)	(52, 01, 32, 61, 21)	(82, 72, 62, 31, 10)		

$(\alpha, \beta) = (8, 6) :$

$Q_1$	(00, 11, 22)	(30, 41, 52)	$Q_2$	(00, 30, 71)	(22, 41, 12)	$Q_3$	(00, 41, 82)	(11, 30, 12)
$P_1$	(00, 52, 60)	(11, 41, 71)	(22, 30, 82)	(90, 20, 21)	(01, 50, 10)			
	(12, 31, 40)	(42, 80, 70)	(61, 62, 92)	(72, 02, 51)	(91, 32, 81)			
$P'_1$	(00, 12, 52, 20, 51)	(11, 50, 71, 70, 02)	(22, 10, 72, 30, 32)					
	(41, 60, 21, 42, 81)	(82, 80, 90, 61, 40)	(01, 91, 62, 31, 92)					

$(\alpha, \beta) = (9, 5) :$

$Q_1$	(00, 11, 22)	(30, 41, 52)	$Q_2$	(00, 30, 71)	(22, 41, 12)	$Q_3$	(00, 41, 82)	(11, 30, 72)
$P_1$	(00, 52, 60)	(11, 41, 71)	(22, 30, 82)	(90, 20, 10)	(01, 50, 40)			
	(12, 31, 80)	(42, 91, 62)	(61, 51, 92)	(72, 21, 81)	(02, 32, 70)			
$P'_1$	(00, 12, 30, 61, 81)	(11, 80, 01, 22, 91)	(41, 50, 71, 10, 70)					
	(52, 42, 51, 72, 92)	(60, 02, 90, 62, 32)	(82, 20, 40, 31, 21)					

$(\alpha, \beta) = (11, 3) :$

$P_1$	(00, 30, 60)	(11, 41, 71)	(22, 52, 82)	(90, 20, 61)	(01, 42, 91)
	(12, 02, 51)	(31, 21, 92)	(50, 40, 81)	(72, 10, 62)	(80, 32, 70)
$P_2$	(00, 71, 31)	(11, 80, 51)	(22, 50, 70)	(30, 61, 10)	(41, 12, 92)
	(52, 90, 62)	(60, 01, 21)	(82, 42, 02)	(20, 72, 40)	(91, 32, 81)
$P'_1$	(00, 42, 62, 22, 51)	(11, 52, 91, 61, 81)	(30, 01, 70, 90, 50)		
	(41, 31, 60, 02, 72)	(71, 10, 20, 92, 32)	(82, 21, 80, 12, 40)		

$(\alpha, \beta) = (12, 2) :$

$P_1$	(00, 11, 22)	(30, 41, 52)	(60, 90, 31)	(71, 01, 20)	(82, 12, 50)
	(42, 61, 40)	(72, 51, 70)	(80, 21, 92)	(91, 32, 62)	(02, 10, 81)
$P_2$	(00, 52, 01)	(11, 72, 80)	(22, 50, 40)	(30, 12, 31)	(41, 42, 32)
	(60, 02, 51)	(71, 10, 62)	(82, 21, 70)	(90, 20, 81)	(61, 91, 92)
$Q_1$	(00, 90, 32)	(11, 01, 42)	$Q_2$	(00, 72, 62)	(11, 90, 21)

$(\alpha, \beta) = (13, 1) :$

$Q_1$	(00, 82, 10)	(11, 01, 32)	$Q_2$	(00, 01, 02)	(11, 72, 10)	$Q_3$	(00, 12, 31)	(22, 01, 10)
$P_1$	(00, 11, 22)	(30, 41, 52)	(60, 90, 20)	(71, 01, 31)	(82, 12, 42)			
	(50, 91, 10)	(61, 80, 32)	(72, 40, 81)	(02, 51, 92)	(21, 62, 70)			
$P_2$	(00, 71, 42)	(11, 02, 21)	(22, 90, 61)	(30, 91, 40)	(41, 01, 80)			
	(52, 60, 51)	(82, 20, 92)	(12, 72, 70)	(31, 10, 62)	(50, 32, 81)			

## Appendix B for Lemma 4.5

$(\alpha, \beta) = (0, 22) : I = \{(i_0, (i+13)_3), (i_1, (i+2)_2) | i \in Z_{15}\}.$

$Q'_1$	(00, 11, 22, 33, 51)	(40, 62, 80, 73, 141)	(102, 03, 60, 63, 20)	(131, 32, 71, 143, 42)
$Q'_2$	(00, 102, 123, 101, 103)	(11, 140, 62, 10, 63)	(22, 83, 131, 70, 71)	(80, 82, 43, 42, 41)

$$\begin{array}{cccc}
T' & (0_0, 12_3, 9_1, 8_2, 6_3) & (0_0, 8_3, 11_1, 9_2, 2_3) & (0_0, 6_2, 3_1, 7_3, 9_2) & (0_0, 9_1, 6_0, 13_1, 12_2) \\
& (0_0, 14_2, 7_1, 13_3, 1_2) & (0_0, 7_3, 1_0, 9_3, 8_1) & (0_0, 11_2, 12_1, 8_0, 4_2) & (0_0, 3_2, 9_0, 12_1, 1_3) \\
& (0_0, 3_1, 14_0, 11_2, 7_1) & (0_0, 13_1, 7_2, 6_0, 14_1) & (0_0, 4_1, 13_2, 6_3, 12_1) & (0_0, 3_3, 2_1, 14_2, 11_3)
\end{array}$$

$(\alpha, \beta) = (6, 16) : I = \{(i_0, (i+4)_2), (i_1, (i+4)_3) | i \in Z_{15}\}.$

$$\begin{array}{cccccc}
T & (0_0, 14_2, 10_1) & (0_0, 10_2, 2_3) & (0_0, 5_3, 4_1) & (0_0, 11_3, 1_2) & (0_0, 13_1, 8_3) & (0_0, 7_1, 5_2) \\
Q'_1 & (0_0, 5_1, 10_2, 3_3, 3_2) & (1_1, 8_0, 0_3, 2_0, 12_3) & & (2_2, 9_1, 9_0, 6_2, 3_1) & & (11_3, 9_2, 14_3, 2_1, 11_0) \\
Q'_2 & (0_0, 2_1, 4_2, 11_0, 11_2) & (1_1, 7_2, 13_3, 5_1, 2_0) & & (4_0, 1_3, 13_2, 13_1, 5_3) & & (7_3, 13_0, 5_2, 4_3, 4_1) \\
T' & (0_0, 6_2, 12_1, 8_2, 4_3) & (0_0, 12_1, 11_0, 14_3, 13_2) & & (0_0, 6_3, 8_1, 2_2, 14_3) & & (0_0, 11_1, 2_3, 3_0, 9_2) \\
& (0_0, 2_2, 6_1, 3_2, 9_1) & (0_0, 1_1, 14_3, 12_1, 3_3) & & & & 
\end{array}$$

$(\alpha, \beta) = (12, 10) : I = \{(i_0, (i+14)_1), (i_2, (i+3)_3) | i \in Z_{15}\}.$

$$\begin{array}{cccccc}
T & (0_0, 1_1, 2_2) & (0_0, 2_3, 13_2) & (0_0, 7_1, 5_2) & (0_0, 8_2, 13_3) & (0_0, 11_3, 4_1) & (0_0, 5_1, 1_2) \\
& (0_0, 7_3, 14_2) & (0_0, 7_2, 14_3) & (0_0, 11_1, 10_3) & (0_0, 2_1, 4_3) & (0_0, 13_1, 8_3) & (0_0, 10_2, 8_1) \\
Q'_1 & (0_0, 3_3, 6_2, 9_1, 0_3) & (1_1, 4_0, 10_2, 1_0, 4_2) & & (2_2, 11_3, 2_1, 8_0, 2_3) & & (5_1, 2_0, 8_1, 3_2, 9_3) \\
Q'_2 & (0_0, 10_1, 4_2, 13_1, 12_2) & (1_1, 12_3, 12_1, 0_3, 1_2) & & (3_3, 12_0, 9_3, 8_0, 8_2) & & (4_0, 0_2, 11_0, 1_3, 4_1)
\end{array}$$

## Appendix C for Lemma 4.6

$(\alpha, \beta) = (18, 11) :$

$$\begin{array}{ccccc}
P & (14_2, 13_3, 7_0) & (0_0, 14_0, 10_3) & (2_2, 4_3, 11_1) & (8_0, 10_1, 12_1) & (3_3, 14_1, 3_0) \\
& (6_2, 8_2, 8_1) & (10_2, 2_0, 13_2) & (9_1, 3_2, 4_2) & (5_0, 6_1, 9_3) & (5_1, 12_3, 10_0) \\
& (13_0, 12_2, 11_0) & (7_3, 0_3, 5_3) & (12_0, 1_2, 6_3) & (4_0, 1_3, 2_3) & (11_3, 7_2, 0_2) \\
& (13_1, 8_3, 3_1) & (2_1, 5_2, 14_3) & (1_1, 9_0, 0_1) & (1_0, 11_2, 6_0) & (7_1, 4_1, 9_2) \\
Q_1 & (0_0, 8_0, 1_3) & (1_1, 12_3, 7_0) & (2_2, 6_1, 5_3) & (5_1, 4_2, 9_2) & \\
Q' & (0_0, 5_1, 6_2, 4_3, 3_1) & (1_1, 2_0, 1_3, 0_2, 3_0) & (2_2, 8_3, 8_2, 0_3, 12_3) & (4_0, 1_0, 4_2, 12_1, 4_1) & \\
T' & (0_0, 2_2, 8_0, 11_3, 4_0) & (0_0, 6_2, 9_1, 2_2, 13_2) & (0_0, 11_0, 3_3, 14_3, 12_1) & (0_0, 7_3, 1_1, 14_3, 8_1) & \\
& (0_0, 3_3, 1_1, 12_1, 9_2) & & & & 
\end{array}$$

$(\alpha, \beta) = (21, 8) :$

$$\begin{array}{ccccc}
P & (10_2, 3_1, 5_2) & (13_1, 7_2, 9_0) & (1_0, 2_0, 12_1) & (2_2, 12_3, 1_2) & (10_1, 10_0, 6_3) \\
& (12_0, 3_2, 1_3) & (1_1, 5_3, 13_3) & (11_2, 6_0, 13_2) & (5_0, 4_2, 11_0) & (2_1, 4_3, 12_2) \\
& (0_3, 2_3, 8_1) & (5_1, 11_1, 9_2) & (4_0, 11_3, 10_3) & (7_3, 9_1, 7_0) & (6_1, 7_1, 14_0) \\
& (6_2, 14_2, 3_0) & (0_0, 8_0, 14_1) & (13_0, 8_2, 14_3) & (3_3, 8_3, 0_1) & (0_2, 9_3, 4_1) \\
Q_1 & (0_0, 2_2, 5_0) & (1_1, 11_1, 2_3) & (3_3, 4_2, 10_0) & (6_2, 0_1, 10_3) & \\
T & (0_0, 4_0, 5_1) & & & & \\
Q' & (0_0, 9_1, 8_2, 8_3, 2_3) & (1_1, 3_1, 10_1, 12_0, 1_2) & (2_2, 4_3, 14_0, 1_0, 10_3) & (8_0, 6_3, 7_1, 0_2, 9_2) & \\
T' & (0_0, 3_3, 6_3, 9_3, 12_3) & (0_0, 12_0, 9_1, 6_2, 3_1) & (0_0, 3_0, 6_3, 9_0, 12_1) & & 
\end{array}$$

$(\alpha, \beta) = (22, 7) :$

$$\begin{array}{ccccc}
P & (2_1, 8_2, 10_0) & (2_2, 9_0, 0_1) & (13_1, 7_2, 14_3) & (14_2, 5_0, 1_2) & (5_1, 1_3, 12_1) \\
& (0_0, 13_0, 12_2) & (11_3, 9_3, 11_0) & (12_3, 0_2, 4_1) & (6_1, 7_1, 5_2) & (8_3, 3_1, 7_0) \\
& (3_3, 1_0, 11_1) & (14_1, 6_0, 13_3) & (8_0, 4_3, 13_2) & (7_3, 3_2, 11_2) & (4_0, 8_1, 10_3) \\
& (12_0, 4_2, 6_3) & (1_1, 2_0, 3_0) & (10_2, 10_1, 5_3) & (6_2, 9_1, 0_3) & (14_0, 2_3, 9_2) \\
Q_1 & (0_0, 2_2, 9_1) & (1_1, 1_0, 13_3) & (3_3, 4_2, 2_3) & (5_1, 8_0, 12_2) & \\
T & (0_0, 4_0, 5_1) & & & & \\
Q' & (0_0, 6_2, 3_1, 1_1, 4_3) & (2_2, 3_2, 2_0, 7_3, 14_0) & (3_3, 11_3, 2_1, 0_3, 0_2) & (5_1, 3_0, 11_0, 14_1, 9_2) & 
\end{array}$$

$(\alpha, \beta) = (23, 6) :$

$$\begin{array}{ccccc}
Q_1 & (0_0, 5_1, 9_1) & (1_1, 8_0, 11_2) & (3_3, 10_2, 10_3) & (4_0, 0_2, 2_3) & \\
S & (0_0, 1_1, 2_2) & (0_0, 4_3, 11_1) & (0_0, 5_0, 7_2) & (0_0, 8_2, 13_2) & (0_0, 10_0, 14_3) \\
Q' & (0_0, 8_0, 3_2, 0_1, 14_2) & (1_1, 6_2, 0_3, 11_3, 13_3) & (2_2, 7_3, 7_0, 7_1, 0_2) & (4_0, 8_1, 6_0, 9_1, 14_3) & 
\end{array}$$

$P$	$(3_1, 13_3, 9_2)$	$(8_3, 7_1, 5_2)$	$(9_1, 5_3, 3_0)$	$(11_2, 6_0, 7_0)$	$(2_2, 1_0, 5_0)$
	$(13_0, 11_1, 12_1)$	$(0_0, 3_3, 6_2)$	$(5_1, 2_1, 10_0)$	$(1_1, 3_2, 14_1)$	$(0_3, 8_2, 14_0)$
	$(8_0, 4_3, 1_3)$	$(12_0, 9_0, 6_3)$	$(4_0, 2_0, 9_3)$	$(7_2, 0_2, 0_1)$	$(11_3, 12_2, 10_3)$
	$(7_3, 1_2, 4_1)$	$(10_1, 4_2, 14_3)$	$(10_2, 14_2, 13_2)$	$(13_1, 6_1, 12_3)$	$(2_3, 8_1, 11_0)$

$(\alpha, \beta) = (24, 5) :$

$P$	$(3_2, 10_1, 14_3)$	$(2_1, 6_0, 0_1)$	$(8_0, 13_1, 3_1)$	$(6_2, 12_3, 11_1)$	$(0_0, 14_0, 10_3)$
	$(1_1, 12_2, 5_2)$	$(4_0, 8_1, 12_1)$	$(2_2, 4_2, 9_3)$	$(10_2, 0_2, 3_0)$	$(1_0, 8_3, 11_2)$
	$(6_1, 7_2, 7_1)$	$(9_1, 5_3, 6_3)$	$(13_0, 1_2, 11_0)$	$(14_2, 1_3, 10_0)$	$(5_1, 4_3, 5_0)$
	$(3_3, 8_2, 9_2)$	$(0_3, 2_0, 7_0)$	$(11_3, 9_0, 13_3)$	$(12_0, 4_1, 13_2)$	$(7_3, 14_1, 2_3)$
$Q_1$	$(0_0, 1_1, 8_2)$	$(3_3, 13_0, 11_1)$	$(6_2, 9_1, 4_3)$	$(8_0, 8_3, 4_2)$	
$Q_2$	$(0_0, 3_3, 12_1)$	$(1_1, 10_0, 14_0)$	$(2_2, 11_1, 13_2)$	$(6_2, 7_3, 14_3)$	
$Q_3$	$(0_0, 6_2, 7_0)$	$(1_1, 6_3, 10_3)$	$(2_2, 8_0, 2_3)$	$(5_1, 10_2, 12_1)$	
$T'$	$(0_0, 2_2, 1_3, 13_2, 14_1)$	$(0_0, 12_3, 9_2, 11_0, 13_2)$	$(0_0, 1_3, 4_0, 2_2, 3_1)$		

$(\alpha, \beta) = (26, 3) :$

$P$	$(3_3, 10_2, 11_0)$	$(5_1, 0_2, 9_3)$	$(2_2, 7_3, 12_0)$	$(0_0, 9_0, 7_0)$	$(0_3, 14_0, 14_3)$
	$(9_1, 2_0, 11_1)$	$(6_2, 8_3, 10_3)$	$(1_1, 10_1, 6_0)$	$(4_0, 12_3, 13_2)$	$(8_0, 2_3, 6_3)$
	$(13_1, 7_2, 14_1)$	$(6_1, 13_0, 8_2)$	$(1_0, 4_2, 12_2)$	$(14_2, 7_1, 5_2)$	$(5_0, 11_2, 9_2)$
	$(2_1, 5_3, 3_0)$	$(3_2, 1_3, 4_1)$	$(11_3, 1_2, 12_1)$	$(4_3, 3_1, 13_3)$	$(10_0, 0_1, 8_1)$
$Q_1$	$(0_0, 4_0, 6_1)$	$(1_1, 6_2, 7_2)$	$(2_2, 9_3, 2_3)$	$(5_1, 5_0, 1_3)$	
$Q_2$	$(0_0, 1_0, 6_3)$	$(1_1, 7_3, 1_2)$	$(2_2, 6_2, 5_0)$	$(5_1, 9_1, 14_3)$	
$S$	$(0_0, 1_1, 2_2)$	$(0_0, 4_3, 11_1)$	$(0_0, 5_0, 7_2)$	$(0_0, 8_2, 13_2)$	$(0_0, 10_0, 14_3)$
$T'$	$(0_0, 3_3, 6_3, 9_3, 12_3)$	$(0_0, 12_0, 9_1, 6_2, 3_1)$	$(0_0, 3_0, 6_3, 9_0, 12_1)$		

$(\alpha, \beta) = (27, 2) :$

$P$	$(5_1, 8_0, 12_0)$	$(0_2, 1_3, 14_3)$	$(8_3, 13_0, 6_0)$	$(10_1, 9_2, 11_0)$	$(4_3, 6_1, 12_3)$
	$(2_0, 1_2, 7_0)$	$(3_3, 4_0, 13_3)$	$(9_1, 9_0, 4_2)$	$(0_0, 1_1, 2_2)$	$(14_1, 9_3, 3_0)$
	$(13_1, 7_2, 11_1)$	$(7_3, 3_1, 5_2)$	$(11_3, 2_1, 12_1)$	$(7_1, 0_1, 13_2)$	$(0_3, 3_2, 10_0)$
	$(5_0, 11_2, 2_3)$	$(6_2, 1_0, 14_0)$	$(10_2, 14_2, 10_3)$	$(5_3, 12_2, 6_3)$	$(8_2, 4_1, 8_1)$
$Q_1$	$(0_0, 3_3, 7_3)$	$(1_1, 2_1, 7_0)$	$(2_2, 5_0, 1_2)$	$(6_2, 9_1, 11_3)$	
$Q_2$	$(0_0, 5_1, 14_0)$	$(1_1, 13_0, 2_3)$	$(2_2, 9_3, 12_1)$	$(6_2, 1_3, 13_2)$	
$Q_3$	$(0_0, 11_3, 4_1)$	$(2_2, 1_0, 4_2)$	$(3_3, 2_0, 0_1)$	$(5_1, 12_2, 10_3)$	
$Q_4$	$(0_0, 0_3, 7_1)$	$(2_2, 7_2, 14_1)$	$(4_0, 6_1, 2_3)$	$(6_2, 2_0, 10_3)$	

$(\alpha, \beta) = (28, 1) :$

$P$	$(0_3, 12_3, 14_1)$	$(3_1, 14_0, 13_2)$	$(0_0, 1_1, 8_1)$	$(2_0, 5_2, 6_3)$	$(2_1, 7_2, 11_0)$
	$(0_2, 12_2, 4_1)$	$(11_3, 6_1, 13_3)$	$(12_0, 1_0, 11_1)$	$(2_2, 10_1, 9_2)$	$(5_1, 8_0, 9_3)$
	$(9_0, 1_3, 7_1)$	$(10_2, 14_2, 14_3)$	$(5_0, 11_2, 3_0)$	$(4_2, 6_0, 0_1)$	$(13_1, 4_3, 8_3)$
	$(7_3, 13_0, 10_0)$	$(6_2, 5_3, 8_2)$	$(9_1, 3_2, 12_1)$	$(3_3, 4_0, 2_3)$	$(1_2, 7_0, 10_3)$
$Q_1$	$(0_0, 2_2, 5_1)$	$(1_1, 13_0, 14_0)$	$(3_3, 7_2, 10_3)$	$(6_2, 6_1, 14_3)$	
$Q_2$	$(0_0, 8_0, 10_3)$	$(1_1, 1_0, 12_3)$	$(2_2, 3_2, 8_3)$	$(5_1, 9_1, 7_2)$	
$Q_3$	$(0_0, 10_2, 7_1)$	$(2_2, 0_3, 10_0)$	$(5_1, 6_1, 8_3)$	$(6_2, 5_0, 13_3)$	
$Q_4$	$(0_0, 0_3, 5_2)$	$(1_1, 14_1, 13_3)$	$(4_0, 0_1, 1_2)$	$(6_2, 8_3, 2_0)$	

## Appendix D for Lemma 5.2

$(\alpha, \beta) = (1, 19) :$

$P'_1$	$(0_0, 1_1, 2_2, 3_0, 4_1, 5_2, 6_0)$	$(7_1, 9_0, 11_2, 8_2, 10_1, 12_0, 0_2)$	$(13_1, 1_0, 5_1, 2_1, 6_2, 3_2, 7_0)$
	$(4_0, 8_1, 13_0, 11_1, 0_1, 9_2, 1_2)$	$(10_0, 2_0, 9_1, 12_2, 8_0, 3_1, 11_0)$	$(4_2, 10_2, 5_0, 12_1, 7_2, 6_1, 13_2)$
$P'_2$	$(0_0, 3_0, 10_1, 1_1, 7_1, 11_2, 2_1)$	$(2_2, 12_0, 4_1, 0_2, 5_2, 8_2, 1_0)$	$(6_0, 9_0, 3_2, 13_1, 4_0, 11_1, 5_1)$
	$(6_2, 1_2, 11_0, 12_2, 6_1, 7_0, 4_2)$	$(8_1, 2_0, 13_2, 3_1, 12_1, 9_2, 7_2)$	$(10_0, 8_0, 0_1, 10_2, 13_0, 5_0, 9_1)$

$P'_3$	(00, 90, 40, 11, 120, 130, 10) (52, 81, 132, 70, 121, 100, 72)	(22, 32, 61, 71, 50, 131, 12) (60, 111, 102, 21, 80, 101, 42)	(30, 122, 41, 51, 31, 82, 01) (112, 92, 110, 02, 91, 62, 20)
$(\alpha, \beta) = (2, 18) :$			
$Q_1$	(00, 30, 71) (22, 52, 101)	$Q_2$	(00, 41, 82) (11, 52, 90)
$P'_1$	(00, 52, 82, 11, 41, 71, 120) (131, 92, 31, 21, 61, 70, 50)		(90, 32, 100, 02, 51, 01, 72) (81, 20, 121, 111, 91, 130, 80)
$P'_2$	(00, 90, 92, 11, 02, 41, 131) (60, 100, 132, 40, 50, 21, 01)		(30, 51, 91, 120, 31, 52, 130) (82, 111, 110, 62, 80, 101, 42)
$T'$	(00, 11, 22, 30, 41, 52, 60) (00, 31, 91, 11, 42, 61, 121)		(00, 22, 41, 101, 120, 11, 132) (00, 112, 52, 21, 130, 102, 80)
$(\alpha, \beta) = (3, 17) :$			
$Q_1$	(00, 11, 22) (30, 41, 52)	$Q_2$	(00, 30, 71) (22, 41, 112)
$Q_3$	(00, 41, 82) (11, 30, 112)		
$P'_1$	(00, 52, 11, 41, 60, 22, 90) (32, 62, 01, 40, 42, 50, 80)		(30, 82, 112, 120, 71, 02, 51) (81, 31, 121, 122, 91, 72, 110)
$P'_2$	(00, 131, 52, 10, 11, 120, 130) (60, 31, 112, 100, 121, 70, 42)		(101, 131, 70, 21, 100, 10, 111) (92, 20, 132, 130, 61, 12, 102)
			(30, 111, 102, 40, 72, 51, 61) (90, 81, 80, 21, 132, 101, 12)
$(\alpha, \beta) = (4, 16) :$			
$Q_1$	(00, 11, 22) (30, 41, 52)	$Q_2$	(00, 30, 71) (22, 41, 112)
$Q_3$	(00, 41, 82) (11, 30, 112)	$Q_4$	(00, 52, 90) (11, 41, 02)
$P'_1$	(00, 60, 11, 52, 22, 30, 101) (51, 122, 61, 92, 42, 70, 20)		(41, 71, 131, 82, 120, 90, 10) (81, 72, 132, 111, 121, 31, 110)
$P'_2$	(00, 112, 131, 22, 120, 01, 10) (60, 92, 102, 70, 80, 62, 72)		(112, 02, 62, 130, 32, 40, 21) (100, 91, 12, 80, 01, 50, 102)
			(30, 81, 31, 90, 111, 101, 12) (02, 61, 121, 122, 110, 32, 91)
$(\alpha, \beta) = (5, 15) :$			
$P'_1$	(00, 11, 22, 30, 41, 52, 82) (32, 81, 130, 51, 100, 50, 121)		(60, 71, 101, 131, 120, 90, 02) (62, 31, 110, 61, 91, 01, 72)
$P'_2$	(00, 112, 22, 10, 60, 131, 50) (52, 62, 91, 02, 80, 51, 20)		(112, 21, 92, 10, 40, 70, 122) (111, 42, 132, 20, 102, 12, 80)
			(41, 100, 72, 81, 132, 40, 122) (90, 12, 101, 110, 21, 31, 102)
$(\alpha, \beta) = (6, 14) :$			
$P_1$	(00, 11, 22) (30, 41, 52) (10, 32, 70) (21, 40, 81) (122, 72, 132) (20, 61, 110)		(60, 71, 101) (82, 112, 131) (51, 62, 92) (100, 130, 12) (31, 91, 102) (50, 80, 121)
$P'_1$	(00, 52, 101, 11, 82, 21, 71) (10, 01, 80, 70, 72, 40, 31)		(22, 90, 41, 120, 30, 112, 32) (62, 12, 121, 122, 110, 92, 50)
$P'_2$	(00, 131, 52, 10, 11, 40, 42) (41, 51, 110, 100, 102, 71, 92)		(60, 131, 51, 111, 02, 100, 61) (81, 42, 132, 20, 102, 130, 91)
			(30, 111, 80, 32, 20, 101, 122) (112, 130, 72, 62, 91, 120, 61)
$(\alpha, \beta) = (7, 13) :$			
$P_1$	(00, 11, 22) (30, 41, 52) (10, 32, 70) (21, 40, 81) (122, 72, 132) (20, 61, 110)		(60, 71, 101) (82, 112, 131) (51, 62, 92) (100, 130, 12) (31, 91, 102) (50, 80, 121)
$P'_1$	(00, 52, 101, 11, 82, 21, 71) (10, 130, 80, 70, 31, 132, 42)		(22, 90, 41, 120, 30, 112, 32) (40, 01, 121, 12, 110, 100, 91)
$P'_2$	(00, 21, 11, 32, 41, 10, 61) (60, 122, 110, 51, 20, 101, 42)		(60, 131, 51, 111, 02, 81, 72) (62, 20, 102, 122, 61, 92, 50)
			(30, 62, 91, 120, 31, 52, 81) (112, 92, 80, 100, 72, 40, 132)
$(\alpha, \beta) = (8, 12) :$			
$T$	(00, 11, 52)		

$$\begin{array}{lll}
P'_1 & (0_0, 2_2, 4_1, 1_1, 3_0, 5_2, 7_1) & (6_0, 8_2, 0_2, 11_2, 13_1, 10_1, 2_1) & (9_0, 12_0, 1_0, 8_1, 3_1, 4_0, 10_0) \\
& (3_2, 12_2, 5_1, 11_1, 4_2, 6_2, 5_0) & (7_0, 13_0, 8_0, 2_0, 9_1, 6_1, 12_1) & (9_2, 7_2, 10_2, 0_1, 13_2, 1_2, 11_0) \\
P'_2 & (0_0, 9_0, 4_0, 3_0, 10_1, 6_2, 13_0) & (1_1, 0_2, 4_1, 5_1, 0_1, 12_0, 11_1) & (2_2, 7_0, 5_0, 6_0, 11_0, 13_1, 4_2) \\
& (5_2, 10_0, 9_1, 8_1, 13_2, 3_2, 1_2) & (7_1, 9_2, 8_0, 12_2, 10_2, 1_0, 3_1) & (8_2, 6_1, 2_1, 12_1, 2_0, 11_2, 7_2)
\end{array}$$

$$(\alpha, \beta) = (9, 11) :$$

$$\begin{array}{lll}
T & (0_0, 2_2, 13_1) & (0_0, 7_1, 3_2) & (0_0, 8_2, 8_1) \\
P'_1 & (0_0, 1_1, 2_2, 3_0, 4_1, 5_2, 6_0) & (7_1, 10_1, 13_1, 8_2, 11_2, 0_2, 9_2) & (9_0, 12_0, 1_0, 5_1, 10_0, 6_2, 11_1) \\
& (2_1, 3_1, 9_1, 4_2, 0_1, 5_0, 11_0) & (3_2, 7_0, 8_0, 13_0, 2_0, 6_1, 12_1) & (4_0, 8_1, 10_2, 1_2, 7_2, 12_2, 13_2) \\
Q' & (0_0, 9_2, 0_1, 5_2, 10_2, 11_2, 13_0) & (1_1, 6_2, 8_0, 9_0, 4_0, 5_0, 9_1) & (3_0, 8_2, 3_1, 11_1, 6_1, 7_2, 5_1)
\end{array}$$

$$(\alpha, \beta) = (10, 10) :$$

$$\begin{array}{llll}
Q_1 & (0_0, 1_1, 2_2) & (3_0, 4_1, 5_2) & Q_2 & (0_0, 3_0, 7_1) & (2_2, 4_1, 11_2) \\
Q_3 & (0_0, 4_1, 9_0) & (1_1, 5_2, 8_2) & Q_4 & (0_0, 5_2, 10_1) & (1_1, 3_0, 10_2) \\
Q_5 & (0_0, 13_1, 2_1) & (2_2, 3_0, 1_2) & & & \\
P'_1 & (0_0, 1_0, 2_2, 5_2, 6_0, 1_1, 13_0) & (3_0, 8_2, 4_1, 7_1, 6_2, 11_0, 7_2) & (9_0, 4_0, 9_1, 0_2, 10_0, 3_2, 5_1) \\
& (10_1, 1_2, 13_1, 9_2, 4_2, 8_0, 5_0) & (11_2, 2_0, 0_1, 7_0, 6_1, 11_1, 12_1) & (12_0, 10_2, 8_1, 3_1, 2_1, 12_2, 13_2)
\end{array}$$

$$(\alpha, \beta) = (11, 9) :$$

$$\begin{array}{lllll}
P_1 & (0_0, 1_1, 2_2) & (3_0, 4_1, 5_2) & (6_0, 7_1, 10_1) & (8_2, 11_2, 13_1) & (9_0, 12_0, 0_2) \\
& (1_0, 3_2, 8_1) & (2_1, 4_0, 9_2) & (5_1, 6_2, 1_2) & (7_0, 12_2, 3_1) & (10_0, 0_1, 9_1) \\
& (11_1, 10_2, 13_2) & (13_0, 2_0, 6_1) & (4_2, 7_2, 11_0) & (5_0, 8_0, 12_1) & \\
P'_1 & (0_0, 7_1, 6_2, 1_1, 10_1, 3_0, 5_1) & (2_2, 11_2, 6_0, 8_2, 7_2, 11_1, 6_1) & (4_1, 2_0, 12_2, 11_0, 1_2, 2_1, 3_1) \\
& (5_2, 10_0, 9_2, 5_0, 4_0, 8_1, 10_2) & (9_0, 4_2, 13_1, 12_1, 13_0, 3_2, 8_0) & (12_0, 7_0, 9_1, 0_2, 13_2, 1_0, 0_1)
\end{array}$$

$$(\alpha, \beta) = (12, 8) :$$

$$\begin{array}{lllll}
P_1 & (0_0, 1_1, 2_2) & (3_0, 4_1, 5_2) & (6_0, 7_1, 10_1) & (8_2, 11_2, 13_1) & (9_0, 12_0, 0_2) \\
& (1_0, 3_2, 7_0) & (2_1, 4_0, 8_1) & (5_1, 6_2, 9_2) & (10_0, 13_0, 1_2) & (11_1, 0_1, 4_2) \\
& (12_2, 7_2, 13_2) & (2_0, 6_1, 11_0) & (3_1, 9_1, 10_2) & (5_0, 8_0, 12_1) & \\
P_2 & (0_0, 5_2, 10_1) & (1_1, 8_2, 2_1) & (2_2, 9_0, 3_2) & (3_0, 11_2, 5_1) & (4_1, 13_1, 6_2) \\
& (6_0, 1_0, 1_2) & (7_1, 13_0, 7_2) & (12_0, 9_2, 9_1) & (0_2, 11_1, 2_0) & (4_0, 3_1, 13_2) \\
& (7_0, 0_1, 8_0) & (8_1, 4_2, 10_2) & (10_0, 5_0, 11_0) & (12_2, 6_1, 12_1) & \\
P'_1 & (0_0, 9_0, 5_1, 4_1, 0_2, 1_1, 9_2) & (2_2, 13_1, 13_0, 5_2, 7_0, 12_1, 1_0) & (3_0, 2_1, 4_2, 7_1, 6_1, 8_2, 0_1) \\
& (6_0, 3_2, 3_1, 12_0, 13_2, 10_0, 7_2) & (10_1, 1_2, 10_2, 4_0, 9_1, 6_2, 2_0) & (11_2, 11_1, 8_0, 12_2, 11_0, 8_1, 5_0)
\end{array}$$

$$(\alpha, \beta) = (13, 7) :$$

$$\begin{array}{lllll}
P_1 & (0_0, 1_1, 2_2) & (3_0, 4_1, 5_2) & (6_0, 7_1, 10_1) & (8_2, 11_2, 13_1) & (9_0, 12_0, 0_2) \\
& (1_0, 3_2, 7_0) & (2_1, 4_0, 8_1) & (5_1, 6_2, 9_2) & (10_0, 13_0, 1_2) & (11_1, 0_1, 4_2) \\
& (12_2, 7_2, 13_2) & (2_0, 6_1, 11_0) & (3_1, 9_1, 10_2) & (5_0, 8_0, 12_1) & \\
P_2 & (0_0, 5_2, 10_1) & (1_1, 8_2, 2_1) & (2_2, 9_0, 3_2) & (3_0, 11_2, 5_1) & (4_1, 13_1, 6_2) \\
& (6_0, 12_0, 1_2) & (7_1, 9_2, 6_1) & (0_2, 12_2, 9_1) & (1_0, 13_0, 7_2) & (4_0, 3_1, 13_2) \\
& (7_0, 0_1, 8_0) & (8_1, 4_2, 10_2) & (10_0, 5_0, 11_0) & (11_1, 2_0, 12_1) & \\
P'_1 & (0_0, 9_0, 7_0, 4_1, 0_2, 1_1, 3_1) & (2_2, 1_0, 5_2, 4_0, 1_2, 10_2, 5_1) & (3_0, 9_2, 6_0, 7_2, 13_1, 8_1, 5_0) \\
& (7_1, 10_0, 9_1, 11_1, 8_2, 6_1, 12_2) & (10_1, 13_0, 12_1, 6_2, 2_0, 11_2, 13_2) & (12_0, 0_1, 11_0, 3_2, 8_0, 2_1, 4_2)
\end{array}$$

$$(\alpha, \beta) = (15, 5) :$$

$$\begin{array}{lllll}
P_1 & (0_0, 1_1, 4_1) & (2_2, 3_0, 6_0) & (5_2, 8_2, 9_0) & (7_1, 10_1, 12_2) & (11_2, 1_0, 6_2) \\
& (12_0, 4_0, 7_0) & (13_1, 0_2, 5_1) & (2_1, 3_1, 7_2) & (3_2, 9_2, 4_2) & (8_1, 0_1, 9_1) \\
& (10_0, 1_2, 5_0) & (11_1, 12_1, 13_2) & (13_0, 2_0, 8_0) & (6_1, 10_2, 11_0) & \\
T & (0_0, 2_2, 13_1) & (0_0, 7_1, 3_2) & (0_0, 8_2, 8_1) & & \\
Q' & (0_0, 6_0, 11_1, 9_0, 0_2, 13_2, 12_2) & (1_1, 10_0, 5_1, 2_1, 11_0, 2_2, 0_1) & (8_2, 3_2, 11_2, 12_0, 3_1, 1_0, 6_1)
\end{array}$$

$$(\alpha, \beta) = (16, 4) :$$

$$Q' \quad (0_0, 9_0, 7_2, 1_0, 6_0, 0_1, 12_0) \quad (1_1, 10_1, 5_1, 11_1, 13_2, 4_0, 9_1) \quad (2_2, 10_0, 5_2, 13_1, 1_2, 3_2, 4_2)$$

$P_1$	$(0_0, 3_0, 6_0)$	$(1_1, 4_1, 7_1)$	$(2_2, 5_2, 0_2)$	$(8_2, 6_2, 7_2)$	$(9_0, 0_1, 8_0)$
	$(10_1, 12_2, 1_2)$	$(11_2, 5_1, 13_2)$	$(12_0, 10_0, 4_2)$	$(13_1, 7_0, 12_1)$	$(1_0, 6_1, 9_1)$
	$(2_1, 11_1, 3_1)$	$(3_2, 9_2, 11_0)$	$(4_0, 13_0, 5_0)$	$(8_1, 2_0, 10_2)$	
$T$	$(0_0, 2_2, 13_1)$	$(0_0, 7_1, 3_2)$	$(0_0, 1_1, 5_2)$		

$(\alpha, \beta) = (17, 3) :$

$P_1$	$(0_0, 1_1, 2_2)$	$(3_0, 6_0, 10_1)$	$(4_1, 7_1, 9_0)$	$(5_2, 8_2, 12_0)$	$(11_2, 13_1, 0_1)$
	$(0_2, 11_1, 13_2)$	$(1_0, 4_2, 9_1)$	$(2_1, 6_2, 5_0)$	$(3_2, 13_0, 8_0)$	$(4_0, 12_2, 6_1)$
	$(5_1, 1_2, 10_2)$	$(7_0, 3_1, 12_1)$	$(8_1, 2_0, 7_2)$	$(9_2, 10_0, 11_0)$	
$S$	$(0_0, 11_2, 3_1)$	$(0_0, 13_1, 6_2)$			

$(\alpha, \beta) = (18, 2) :$

$P_1$	$(0_0, 1_1, 2_2)$	$(3_0, 6_0, 10_1)$	$(4_1, 7_1, 9_0)$	$(5_2, 8_2, 12_0)$	$(11_2, 13_1, 6_2)$
	$(0_2, 1_2, 9_1)$	$(1_0, 7_0, 6_1)$	$(2_1, 5_0, 13_2)$	$(3_2, 0_1, 8_0)$	$(4_0, 13_0, 10_2)$
	$(5_1, 2_0, 7_2)$	$(8_1, 12_2, 4_2)$	$(9_2, 10_0, 11_0)$	$(11_1, 3_1, 12_1)$	
$S$	$(0_0, 10_1, 4_2)$	$(0_0, 2_1, 1_2)$	$(0_0, 3_2, 8_1)$		

$(\alpha, \beta) = (19, 1) :$

$P_1$	$(0_0, 3_0, 7_1)$	$(1_1, 4_1, 8_2)$	$(2_2, 5_2, 9_0)$	$(6_0, 11_2, 9_2)$	$(10_1, 7_2, 13_2)$
	$(12_0, 6_2, 11_0)$	$(13_1, 7_0, 2_0)$	$(0_2, 10_0, 8_0)$	$(1_0, 12_2, 6_1)$	$(2_1, 11_1, 3_1)$
	$(3_2, 4_2, 9_1)$	$(4_0, 0_1, 12_1)$	$(5_1, 1_2, 10_2)$	$(8_1, 13_0, 5_0)$	

## Appendix E for Lemma 5.3

$(\alpha, \beta) = (0, 31) : I = \{(i_0, (i+17)_2), (i_1, (i+17)_3) | i \in Z_{21}\}.$

$Q'_1$	$(0_0, 2_2, 4_0, 1_1, 3_3, 5_1, 10_2)$	$(6_2, 9_1, 15_3, 8_0, 11_3, 14_2, 19_3)$	$(7_3, 12_0, 17_1, 3_0, 13_1, 1_2, 16_0)$
	$(18_2, 4_1, 6_0, 2_3, 7_1, 5_2, 13_3)$		
$Q'_2$	$(0_0, 9_1, 3_3, 10_2, 16_0, 1_1, 11_3)$	$(2_2, 12_0, 4_1, 4_0, 18_2, 5_1, 19_3)$	$(6_2, 15_3, 5_2, 7_3, 17_1, 15_0, 9_3)$
	$(13_1, 0_2, 20_0, 7_1, 3_0, 8_2, 20_3)$		
$Q'_3$	$(0_0, 17_1, 2_2, 13_1, 10_3, 1_1, 18_2)$	$(4_0, 2_3, 15_0, 5_1, 5_2, 11_3, 13_2)$	$(7_3, 3_0, 10_2, 19_0, 19_3, 16_1, 7_2)$
	$(15_3, 2_0, 1_2, 11_1, 20_0, 13_3, 14_1)$		
$Q'_4$	$(0_0, 2_3, 11_1, 4_0, 14_3, 9_1, 4_2)$	$(1_1, 5_2, 10_0, 13_1, 13_3, 8_0, 5_3)$	$(2_2, 1_3, 15_2, 12_0, 7_1, 10_2, 3_1)$
	$(3_3, 2_0, 11_3, 0_2, 19_1, 20_0, 20_2)$		
$T'$	$(0_0, 1_1, 2_2, 3_3, 4_0, 19_1, 13_2)$	$(0_0, 6_3, 19_1, 18_0, 3_3, 9_2, 15_1)$	$(0_0, 8_2, 16_0, 3_2, 11_0, 19_2, 20_3)$

$(\alpha, \beta) = (9, 22) : I = \{(i_0, (i+8)_2), (i_1, (i+4)_3) | i \in Z_{21}\}.$

$Q_1$	$(0_0, 6_2, 13_1)$	$(2_2, 8_0, 15_3)$	$(4_0, 19_3, 3_1)$	$(5_1, 11_3, 1_2)$	
$Q_2$	$(0_0, 9_1, 18_2)$	$(1_1, 8_0, 19_3)$	$(2_2, 11_3, 17_1)$	$(3_3, 10_2, 19_0)$	
$Q_3$	$(0_0, 10_2, 0_1)$	$(1_1, 11_3, 7_0)$	$(2_2, 19_3, 8_1)$	$(3_3, 18_2, 2_0)$	
$Q'_1$	$(0_0, 14_2, 1_1, 12_0, 2_2, 16_0, 12_1)$ $(13_1, 5_3, 0_1, 12_2, 20_0, 20_3, 8_2)$		$(3_3, 17_1, 11_0, 9_1, 2_3, 15_0, 3_2)$		$(6_2, 3_0, 15_3, 4_2, 7_3, 4_1, 4_3)$
$Q'_2$	$(0_0, 2_3, 8_2, 1_1, 10_3, 4_0, 14_3)$ $(8_0, 5_3, 9_1, 12_2, 13_1, 11_2, 11_1)$		$(2_2, 12_1, 5_0, 0_1, 10_0, 10_2, 6_0)$		$(6_2, 1_3, 10_1, 18_3, 9_0, 14_2, 13_3)$
$T'$	$(0_0, 1_1, 19_2, 6_3, 10_0, 11_2, 16_3)$ $(0_0, 13_3, 15_0, 11_3, 3_2, 5_0, 2_1)$ $(0_0, 3_3, 2_1, 1_0, 6_1, 11_2, 19_3)$		$(0_0, 2_2, 4_3, 6_1, 3_2, 5_3, 1_2)$ $(0_0, 4_1, 5_2, 10_3, 15_0, 2_1, 20_2)$ $(0_0, 8_1, 10_3, 2_2, 19_1, 20_2, 18_1)$		$(0_0, 17_3, 18_0, 15_1, 16_3, 20_0, 19_2)$ $(0_0, 5_1, 4_3, 2_1, 3_2, 1_1, 20_3)$

$(\alpha, \beta) = (18, 13) : I = \{(i_0, (i+3)_1), (i_2, (i+12)_3) | i \in Z_{21}\}.$

$Q_1$	$(0_0, 2_2, 5_1)$	$(1_1, 3_3, 6_2)$	$(4_0, 7_3, 0_1)$	$(8_0, 1_2, 5_3)$
$Q_2$	$(0_0, 7_3, 9_1)$	$(1_1, 4_0, 1_2)$	$(2_2, 11_3, 20_0)$	$(3_3, 18_2, 8_1)$
$Q_3$	$(0_0, 10_2, 15_3)$	$(1_1, 8_0, 19_3)$	$(2_2, 4_0, 12_1)$	$(5_1, 9_2, 5_3)$
$Q_4$	$(0_0, 19_3, 5_2)$	$(1_1, 10_2, 20_0)$	$(3_3, 0_1, 10_0)$	$(5_1, 14_3, 12_2)$
$Q_5$	$(0_0, 0_1, 10_3)$	$(1_1, 15_3, 17_2)$	$(4_0, 8_1, 4_2)$	$(6_2, 14_3, 5_0)$

$Q_6$	$(0_0, 2_3, 12_1)$	$(1_1, 16_0, 4_2)$	$(2_2, 17_1, 18_3)$	$(6_2, 20_0, 16_3)$	
$Q'_1$	$(0_0, 16_1, 7_3, 12_0, 1_1, 3_0, 20_2)$	$(4_0, 8_2, 11_3, 6_0, 19_3, 19_2, 11_1)$	$(2_2, 0_1, 14_2, 2_0, 6_3, 10_1, 9_3)$	$(3_3, 20_1, 11_2, 1_3, 1_0, 17_2, 19_1)$	
$T'$	$(0_0, 13_2, 12_3, 11_0, 3_2, 9_1, 8_2)$	$(0_0, 6_3, 19_0, 18_3, 17_2, 16_1, 1_3)$	$(0_0, 1_1, 9_0, 17_2, 11_3, 19_2, 20_3)$	$(0_0, 20_1, 5_0, 11_3, 3_0, 16_1, 8_3)$	$(0_0, 13_1, 5_3, 18_1, 17_2, 16_3, 15_2)$
$(\alpha, \beta) = (24, 7) : I = \{(i_0, (i+1)_3), (i_1, (i+20)_2)   i \in Z_{21}\}.$					
$Q_1$	$(0_0, 1_1, 2_2)$	$(3_3, 4_0, 9_1)$	$(5_1, 7_3, 10_2)$	$(6_2, 8_0, 11_3)$	
$Q_2$	$(0_0, 6_2, 7_3)$	$(1_1, 4_0, 14_2)$	$(3_3, 5_1, 20_0)$	$(9_1, 2_3, 13_2)$	
$Q_3$	$(0_0, 9_1, 15_3)$	$(1_1, 8_0, 1_2)$	$(2_2, 5_1, 2_3)$	$(4_0, 9_2, 1_3)$	
$Q_4$	$(0_0, 11_3, 18_2)$	$(1_1, 10_2, 20_0)$	$(2_2, 9_1, 19_3)$	$(3_3, 16_0, 8_1)$	
$Q_5$	$(0_0, 17_1, 13_2)$	$(1_1, 2_3, 7_0)$	$(2_2, 8_0, 6_3)$	$(6_2, 0_1, 13_3)$	
$Q_6$	$(0_0, 0_1, 5_3)$	$(1_1, 9_2, 18_3)$	$(2_2, 16_0, 20_1)$	$(7_3, 1_2, 14_0)$	
$Q_7$	$(0_0, 1_2, 12_1)$	$(1_1, 10_3, 19_0)$	$(2_2, 20_0, 5_3)$	$(3_3, 17_1, 12_2)$	
$Q_8$	$(0_0, 2_3, 20_1)$	$(1_1, 1_3, 20_2)$	$(3_3, 7_0, 16_2)$	$(6_2, 12_1, 2_0)$	
$Q'_1$	$(0_0, 8_1, 7_3, 17_1, 19_0, 5_1, 17_2)$	$(6_2, 5_3, 11_2, 0_1, 10_0, 5_2, 6_0)$	$(2_2, 11_0, 20_3, 1_2, 20_1, 3_3, 16_1)$	$(8_0, 18_3, 7_2, 2_3, 11_1, 16_0, 8_3)$	

## Appendix F for Lemma 5.4

$(\alpha, \beta) = (36, 5) :$

$P$	$(0_0, 16_2, 6_3)$	$(3_0, 10_0, 2_2)$	$(19_1, 8_2, 12_2)$	$(4_0, 16_1, 10_2)$	$(12_1, 5_3, 16_3)$	$(9_2, 17_2, 12_3)$
	$(7_0, 11_0, 15_1)$	$(20_0, 18_1, 0_3)$	$(2_1, 10_1, 18_3)$	$(0_2, 2_3, 15_3)$	$(5_1, 7_1, 18_2)$	$(5_0, 15_0, 3_3)$
	$(13_0, 13_2, 15_2)$	$(17_0, 7_3, 9_3)$	$(16_0, 13_1, 4_2)$	$(1_0, 3_1, 4_3)$	$(11_1, 1_3, 8_3)$	$(18_0, 1_2, 11_2)$
	$(12_0, 19_2, 19_3)$	$(6_0, 14_0, 20_1)$	$(2_0, 9_1, 5_2)$	$(0_1, 4_1, 14_1)$	$(8_0, 17_1, 20_2)$	$(7_2, 14_2, 13_3)$
	$(6_1, 3_2, 11_3)$	$(1_1, 6_2, 20_3)$	$(9_0, 8_1, 17_3)$	$(19_0, 10_3, 14_3)$		
$Q_1$	$(1_1, 7_2, 16_3)$	$(1_0, 2_1, 6_2)$	$(0_1, 8_2, 0_3)$	$(0_0, 2_0, 2_3)$		
$Q_2$	$(0_0, 16_1, 18_2)$	$(1_0, 16_2, 5_3)$	$(2_0, 2_1, 19_3)$	$(0_1, 20_2, 6_3)$		
$Q_3$	$(2_0, 15_1, 1_3)$	$(0_0, 10_2, 15_3)$	$(1_0, 16_1, 2_2)$	$(2_1, 18_2, 14_3)$		
$Q_4$	$(2_0, 19_2, 16_3)$	$(0_0, 3_1, 5_3)$	$(1_0, 11_1, 9_2)$	$(1_1, 2_2, 0_3)$		
$Q_5$	$(0_0, 11_1, 11_2)$	$(1_0, 6_1, 19_3)$	$(2_0, 0_2, 12_3)$	$(1_1, 10_2, 11_3)$		
$T'$	$(0_0, 1_1, 2_2, 3_3, 4_0, 5_1, 6_2)$	$(0_0, 15_3, 9_1, 3_3, 19_3, 13_1, 18_2)$	$(0_0, 3_3, 6_2, 1_1, 4_0, 19_3, 16_0)$	$(0_0, 5_1, 8_0, 3_3, 18_2, 2_2, 20_0)$		

$(\alpha, \beta) = (37, 4) :$

$P$	$(0_0, 19_2, 0_3)$	$(5_1, 15_1, 9_3)$	$(3_0, 14_1, 12_3)$	$(0_1, 8_1, 17_1)$	$(6_0, 14_0, 19_1)$	$(13_2, 10_3, 20_3)$
	$(5_0, 9_1, 5_2)$	$(15_0, 8_2, 12_2)$	$(2_2, 11_2, 11_3)$	$(3_2, 16_2, 1_3)$	$(7_0, 17_0, 4_1)$	$(20_0, 6_1, 13_3)$
	$(16_1, 1_2, 15_3)$	$(12_0, 13_1, 16_3)$	$(16_0, 6_3, 19_3)$	$(2_0, 18_2, 7_3)$	$(18_1, 14_3, 18_3)$	$(1_1, 20_1, 20_2)$
	$(3_1, 0_2, 8_3)$	$(11_1, 2_3, 4_3)$	$(13_0, 2_1, 15_2)$	$(4_0, 8_0, 3_3)$	$(9_0, 18_0, 10_2)$	$(7_1, 6_2, 17_2)$
	$(1_0, 10_1, 4_2)$	$(10_0, 12_1, 14_2)$	$(19_0, 7_2, 9_2)$	$(11_0, 5_3, 17_3)$		
$Q_1$	$(0_1, 7_2, 6_3)$	$(1_0, 20_0, 11_3)$	$(0_0, 19_1, 6_2)$	$(2_1, 14_2, 10_3)$		
$Q_2$	$(0_1, 16_2, 11_3)$	$(0_0, 5_2, 18_3)$	$(1_0, 7_1, 18_2)$	$(2_0, 17_1, 19_3)$		
$Q_3$	$(1_0, 0_2, 3_3)$	$(2_0, 1_1, 10_3)$	$(2_1, 5_2, 20_3)$	$(0_0, 3_1, 7_2)$		
$Q_4$	$(1_0, 1_1, 2_3)$	$(0_1, 9_2, 13_3)$	$(0_0, 17_1, 10_2)$	$(2_0, 14_2, 15_3)$		
$Q_5$	$(0_0, 14_1, 15_2)$	$(1_0, 13_1, 8_3)$	$(2_0, 10_2, 0_3)$	$(0_1, 5_2, 10_3)$		
$T'$	$(0_0, 1_1, 2_2, 3_3, 4_0, 5_1, 6_2)$	$(0_0, 15_3, 9_1, 3_3, 19_3, 13_1, 18_2)$	$(0_0, 3_3, 6_2, 1_1, 4_0, 19_3, 16_0)$	$(0_0, 5_1, 8_0, 3_3, 18_2, 2_2, 20_0)$		



$(\alpha, \beta) = (38, 3) :$

$P$	$(3_1, 5_2, 19_3)$ $(16_0, 12_1, 13_3)$ $(10_0, 0_1, 8_3)$ $(8_1, 2_2, 14_3)$ $(2_0, 16_2, 16_3)$	$(17_0, 5_1, 12_3)$ $(3_0, 9_2, 10_2)$ $(6_1, 14_1, 5_3)$ $(16_1, 1_2, 4_3)$ $(8_0, 11_1, 15_3)$	$(6_0, 13_1, 10_3)$ $(18_1, 7_2, 18_3)$ $(15_0, 20_2, 0_3)$ $(0_0, 11_2, 20_3)$ $(4_0, 19_1, 14_2)$	$(7_1, 6_2, 19_2)$ $(5_0, 13_2, 17_3)$ $(1_0, 9_0, 0_2)$ $(17_2, 2_3, 3_3)$ $(19_0, 1_3, 9_3)$	$(18_0, 9_1, 12_2)$ $(14_0, 20_1, 18_2)$ $(7_0, 17_1, 3_2)$ $(4_1, 4_2, 6_3)$	$(1_1, 2_1, 15_2)$ $(12_0, 13_0, 10_1)$ $(11_0, 15_1, 11_3)$ $(20_0, 8_2, 7_3)$
$Q_1$	$(0_0, 0_1, 1_2)$	$(1_0, 17_1, 10_3)$	$(2_0, 5_2, 3_3)$	$(1_1, 18_2, 14_3)$		
$Q_2$	$(0_0, 8_1, 19_2)$	$(1_0, 6_1, 16_3)$	$(2_0, 18_2, 15_3)$	$(1_1, 5_2, 20_3)$		
$Q_3$	$(0_0, 13_1, 18_2)$	$(1_0, 3_1, 18_3)$	$(2_0, 2_2, 7_3)$	$(2_1, 10_2, 5_3)$		
$Q_4$	$(0_0, 14_1, 2_2)$	$(1_0, 0_1, 11_3)$	$(2_0, 15_2, 4_3)$	$(1_1, 19_2, 6_3)$		
$T$	$(0_0, 7_3, 11_3)$	$(0_0, 14_2, 19_3)$	$(0_0, 5_1, 16_0)$	$(0_0, 2_2, 4_0)$	$(0_0, 10_2, 17_1)$	

$(\alpha, \beta) = (39, 2) :$

$P$	$(13_1, 10_2, 17_2)$ $(6_2, 3_3, 14_3)$ $(11_2, 15_2, 10_3)$ $(11_0, 20_2, 12_3)$ $(6_0, 2_2, 17_3)$	$(4_0, 7_1, 20_1)$ $(19_0, 11_1, 16_2)$ $(16_0, 2_3, 19_3)$ $(5_0, 19_2, 9_3)$ $(4_1, 6_1, 13_3)$	$(0_0, 7_2, 0_3)$ $(1_0, 17_0, 0_2)$ $(2_0, 15_0, 9_1)$ $(12_0, 0_1, 18_1)$ $(1_1, 18_2, 16_3)$	$(10_1, 14_1, 5_3)$ $(12_1, 17_1, 15_3)$ $(14_0, 5_2, 8_2)$ $(8_0, 9_2, 14_2)$ $(19_1, 12_2, 13_2)$	$(20_0, 4_3, 18_3)$ $(3_0, 1_2, 3_2)$ $(18_0, 8_1, 15_1)$ $(7_0, 9_0, 10_0)$	$(13_0, 2_1, 1_3)$ $(3_1, 7_3, 20_3)$ $(6_3, 8_3, 11_3)$ $(5_1, 16_1, 4_2)$
$Q_1$	$(1_1, 2_1, 8_2)$	$(1_0, 8_0, 7_3)$	$(0_0, 0_1, 2_3)$	$(0_2, 10_2, 6_3)$		
$Q_2$	$(1_1, 11_3, 12_3)$	$(0_0, 14_1, 3_2)$	$(0_1, 11_2, 19_2)$	$(1_0, 5_0, 13_3)$		
$Q_3$	$(0_1, 2_2, 6_3)$	$(0_0, 16_2, 17_3)$	$(1_1, 9_2, 19_3)$	$(1_0, 11_0, 2_1)$		
$Q_4$	$(2_0, 10_1, 10_2)$	$(1_0, 3_1, 16_3)$	$(0_0, 11_2, 18_3)$	$(2_1, 18_2, 2_3)$		
$Q_5$	$(0_0, 17_1, 10_3)$	$(1_0, 14_2, 17_3)$	$(2_0, 0_1, 12_2)$	$(1_1, 4_2, 6_3)$		
$Q_6$	$(0_0, 4_1, 5_2)$	$(1_0, 6_1, 14_3)$	$(2_0, 4_2, 16_3)$	$(2_1, 15_2, 3_3)$		

$(\alpha, \beta) = (40, 1) :$

$P$	$(13_1, 0_2, 16_2)$ $(13_2, 15_2, 12_3)$ $(9_0, 1_2, 9_2)$ $(2_0, 17_0, 11_1)$ $(2_1, 5_3, 11_3)$	$(10_0, 3_2, 7_3)$ $(4_2, 4_3, 14_3)$ $(19_1, 8_2, 11_2)$ $(4_0, 8_1, 18_1)$ $(20_0, 7_2, 9_3)$	$(15_1, 20_1, 10_2)$ $(3_0, 5_2, 20_2)$ $(14_1, 16_1, 18_3)$ $(7_0, 6_2, 20_3)$ $(15_0, 4_1, 12_1)$	$(5_0, 8_0, 14_2)$ $(0_1, 3_1, 15_3)$ $(0_3, 1_3, 19_3)$ $(12_0, 8_3, 13_3)$ $(10_1, 17_2, 2_3)$	$(13_0, 2_2, 17_3)$ $(1_1, 7_1, 6_3)$ $(18_0, 5_1, 9_1)$ $(0_0, 18_2, 19_2)$	$(1_0, 3_3, 16_3)$ $(19_0, 17_1, 10_3)$ $(6_0, 14_0, 16_0)$ $(11_0, 6_1, 12_2)$
$Q_1$	$(1_0, 17_0, 12_3)$	$(1_1, 2_1, 6_2)$	$(0_0, 6_1, 7_3)$	$(1_2, 5_2, 17_3)$		
$Q_2$	$(0_1, 6_3, 10_3)$	$(0_0, 17_0, 5_3)$	$(1_0, 8_1, 5_2)$	$(1_1, 0_2, 10_2)$		
$Q_3$	$(1_0, 12_2, 4_3)$	$(0_0, 20_0, 20_1)$	$(1_1, 2_2, 9_3)$	$(0_1, 19_2, 17_3)$		
$Q_4$	$(1_0, 12_1, 8_2)$	$(2_0, 18_2, 0_3)$	$(0_0, 17_1, 14_3)$	$(1_1, 13_2, 1_3)$		
$Q_5$	$(2_0, 14_2, 1_3)$	$(0_1, 15_2, 11_3)$	$(1_0, 4_1, 4_2)$	$(0_0, 2_1, 0_3)$		
$Q_6$	$(0_0, 1_1, 15_2)$	$(1_0, 14_1, 9_3)$	$(2_0, 7_2, 8_3)$	$(0_1, 2_2, 7_3)$		